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Plimpton 322 is Babylonian exact sexagesimal trigonometry

Daniel F. Mansfield, N.J. Wildberger

School of Mathematics and Statistics, UNSW, Sydney, Australia

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Abstract

We trace the origins of trigonometry to the Old Babylonian era, between the 19th and 16th centuries B.C.E. This is well over a millennium before Hipparchus is said to have fathered the subject with his 'table of chords'. The main piece of evidence comes from the most famous of Old Babylonian tablets: Plimpton 322, which we interpret in the context of the Old Babylonian approach to triangles and their preference for numerical accuracy. By examining the evidence with this mindset, and comparing Plimpton 322 with Madhava's table of sines, we demonstrate that Plimpton 322 is a powerful, exact ratio-based trigonometric table.

MSC: 01A17

Keywords: Plimpton 322; Trigonometry; Old Babylonian; Tablet; Pythagorean triples

1. Introduction

Plimpton 322 (P322) is one of the most sophisticated scientific artifacts of the ancient world, containing 15 rows of arithmetically complicated Pythagorean triples. But the purpose of this table has mostly eluded scholars, despite intense investigation.

We argue that the numerical complexity of P322 proves that it is not a scribal school text, as many authors have claimed. Instead, P322 is a trigonometric table of a completely unfamiliar kind and was ahead of its time by thousands of years.

To see how, we must adopt two ideas that are unique to the mathematical culture of the Old Babylonian (OB) period, between the 19th and 16th centuries B.C.E.

First we abandon the notion of angle, and instead describe a right triangle in terms of the short side, long side and diagonal of a rectangle. Second we must adopt the OB number system and its emphasis on precision. The OB scribes used a richer sexagesimal (base 60) system which is more suitable for exact

E-mail address: daniel.mansfield@unsw.edu.au (D.F. Mansfield).

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computation than our decimal system, and while they were not shy of approximation they had a preference for exact calculation.

A modern trigonometric table is a list of right triangles with hypotenuse 1 and approximations to the side lengths $\sin\theta$ and $\cos\theta$, along with the ratio $\tan\theta = \sin\theta/\cos\theta$. We propose that P322 is a different kind of trigonometric table which lists right triangles with long side 1, exact short side β and exact diagonal δ – in place of the approximations $\sin\theta$ and $\cos\theta$. The ratios β/δ or δ/β (equivalent to $\tan\theta$) are not given because they cannot be calculated exactly on account of the divisions involved. Instead P322 separates this information into three exact numbers: a related squared ratio which can be used as an index, and simplified values b and d for β and δ which allow the user to make their own approximation to these ratios.

If this interpretation is correct, then P322 replaces Hipparchus' 'table of chords' as the world's *oldest trigonometric table* — but it is additionally unique because of its exact nature, which would make it the world's only *completely accurate* trigonometric table. These insights expose an entirely new level of sophistication for OB mathematics.

We present an improved approach to the generation and reconstruction of the table which concurs with Britton, Proust and Shnider (2011) on the likely missing columns. We present the generally accepted reconstruction of the table both in sexagesimal form as P322(CR) and also in approximate decimal form as P322(CR-Decimal8). We show that in principle the information on P322(CR) is sufficient to perform the same function as a modern trigonometric table using only OB techniques, and we apply it to contemporary OB questions regarding the measurements of a rectangle.

We then exhibit the impressive mathematical power of P322(CR-Decimal8) by showing that P322 holds its own as a computational device even against Madhava's sine table from 3000 years later. This is a strong argument that the essential purpose of P322 was indeed trigonometric: suggesting that an OB scribe unwittingly created an effective trigonometric table 3000 years ahead of its time is an untenable position.

The novel approach to trigonometry and geometrical problems encapsulated by P322 resonates with modern investigations centered around *rational trigonometry* both in the Euclidean and non-Euclidean settings, including both hyperbolic and elliptic or spherical geometries (Wildberger, 2005, 2013, 2010). The classical framing of trigonometry and geometry around circles and angular measurement is only one of a spectrum of possible approaches, so perhaps we should view angular trigonometry as a social construct originating from the needs of Seleucid astronomy rather than a necessary and intrinsic aspect of geometry.

The paper is laid out as follows. In the first section we introduce P322 itself. Then we review contemporary geometric concepts: the roles of reciprocal slope, known as *ukullû* or *indanum*, properties of similar triangles and the Diagonal rule. This is followed by a review of OB numerical procedures: the role of reciprocal tables, the square side rule, the square root algorithm and the trailing part algorithm. We then return to P322 to discuss its method of construction, missing columns and unfilled rows.

Then we turn our attention to the main thrust of the paper, which is to investigate the trigonometric interpretation of P322. We begin by critiquing current views as to the purpose of P322 and give our own novel interpretation of the tablet. Our comparison of the power of P322(CR-Decimal8) with Madhava's sine table brings out also the important role of the mysterious column I', and shows possible uses for the entries of II' and III'. Finally we follow Knuth (1972) in showing that the OB tablet AO 6770 demonstrates an understanding of linear interpolation, which increases the potential range and power of P322 considerably.

Further research is required to investigate the historical and mathematical possibilities we are suggesting. On the historical side the question arises of how the Babylonians might have used such a table, and we do not attempt to answer this question here. On the mathematical side it is becoming increasingly clear that the OB tradition of step-by-step procedures based on their concrete and powerful arithmetical system is much richer than we formerly imagined. Perhaps the understanding of this ancient culture can help inspire new directions in modern mathematics and education.

Figure 1. Plimpton 322: The world's first trigonometric table, courtesy of the Rare Book and Manuscript Library, Columbia University.

2. Plimpton 322

Ever since its publication in the classic *Mathematical Cuneiform Texts* by Neugebauer and Sachs (1945), modern scholars have been fascinated by P322 (Figure 1) on account of its connection with Pythagorean triples.

Edgar Banks sold the tablet to G.A. Plimpton around 1922 and originally attributed its provenance to Larsa, an ancient Sumerian city near the Persian Gulf. By comparing the style of the script to other OB texts from the region, Robson (2001, 172) has dated it to between 1822 and 1762 B.C.E., which is around the time of Hammurabi.

Physically the tablet is made from clay and measures 12.7 cm by 8.8 cm, with the left-hand edge showing clear evidence of being broken, and indeed remnants of modern glue suggests that the break occurred in recent times.

The obverse (front) is divided by three vertical lines into four columns, each with a heading, the first of which is partially obscured by damage, while the others are clearly readable. The main body of the obverse is ruled by neat horizontal lines into fifteen equally spaced rows containing sexagesimal numbers, some of which are quite large. The vertical lines continue on the bottom and reverse, which are otherwise empty.

Numbers in the OB sexagesimal place value system were written as a finite sequence of digits, and we use our numbers 1 to 59 to represent the OB digits, separated by a period "." to denote the relative place value. In OB times zero was usually represented by a space, and the separation between the integer and fractional part of a number was determined by the context. To represent these concepts we adopt the convention of using "0" for "zero" and ";" for the radix, or sexagesimal point, separating the integral and fraction part of a number when the context makes this clear.

There are several errors in the tablet, due either to calculation or copying mistakes. These errors have been the subject of intense study (Gillings, 1953; Abdulaziz, 2010; Britton et al., 2011; Buck, 1980). Table 1

Table 1

$I' : \delta_n^2$	$II' : b_n$	$III' : d_n$	$IV' : n$
The takiltum of the diagonal from which 1 *is subtracted and that of the width comes up*	*ib-si of the width*	*ib-si of the diagonal*	*its place*
1.59.00.15	1.59	2.49	*ki* 1
1.56.56.58.14.50.06.15	56.07	1.20.25	*ki* 2
(1.56.56.58.14.56.15)		(3.12.01)	
1.55.07.41.15.33.45	1.16.41	1.50.49	*ki* 3
1.53.10.29.32.52.16	3.31.49	5.09.01	*ki* 4
1.48.54.01.40	1.05	1.37	*ki* 5
1.47.06.41.40	5.19	8.01	*ki* 6
1.43.11.56.28.26.40	38.11	59.01	*ki* 7
1.41.33.45.14.03.45	13.19	20.49	*ki* 8
(1.41.33.59.03.45)			
1.38.33.36.36	8.01	12.49	*ki* 9
	(9.01)		
1.35.10.02.28.27.24.26.40	1.22.41	2.16.01	*ki* 10
1.33.45	45	1.15	*ki* 11
1.29.21.54.02.15	27.59	48.49	*ki* 12
1.27.00.03.45	2.41	4.49	*ki* 13
	(7.12.01)		
1.25.48.51.35.06.40	29.31	53.49	*ki* 14
1.23.13.46.40	28	53	*ki* 15
	(56)		

shows what is generally agreed to be the correct contents of P322, with errors in round brackets displayed below the corrections; and an additional top row showing our labeling of the columns' contents above the headings. The columns are traditionally referred to as I', II', III' and IV'—where the prime acknowledges that this column numbering is a modern addition which may differ from the original.

The numbers in this table are quite remarkable, as Neugebauer and Sachs (1945) revealed to an astonished mathematical world. Let us summarize the contents by column.

The original translation of the damaged column I' heading has been successively improved upon by Robson (2001, 192) and most recently Britton et al. (2011, 526), who translate it as:

> The *takiltum* of the diagonal (from) which 1 is subtracted and (that of) the width comes up.

The word *takiltum* is difficult to interpret, and it may just mean square (Thureau-Dangin, 1937, 23). In keeping with this we denote the corresponding entry in row n by δ_n^2, but this is no ordinary square! The heading indicates that if we subtract the leading 1 then we get another square, namely $\delta_n^2 - 1 = \beta_n^2$. Hence both β_n^2 and δ_n^2 are made explicit by the heading and must be relevant to the tablets purpose. Some scholars have argued that the leading 1's which are obscured by the break are not actually there, and that the column instead contains β_n^2. Closer inspection has shown, in agreement with the column heading, that the 1's were originally there and likely contributed to the break (Robson, 2001, 191; Britton et al., 2011, 524). We will interpret the contents as numbers with a sexagesimal point directly after the leading 1, so that the corresponding β_n^2 values are numbers between 0 and 1, but we should be mindful that in OB arithmetic relative magnitudes are always subject to interpretation.

The column II' and III' headings are much simpler. They are the *ib-si of the width* and *ib-si of the diagonal* respectively, where the word *ib-si* refers to the result of some operation. We denote these entries as b_n and d_n respectively, and together with the implicit long side l_n they also form the sides of a rectangle or right triangle satisfying $b^2 + l^2 = d^2$. Furthermore these numbers are related to the entry in column I'

by:

$$\delta_n^2 = \left(\frac{d_n}{l_n}\right)^2, \quad \beta_n^2 = \left(\frac{b_n}{l_n}\right)^2.$$

Each of the 15 values of l_n have the form $2^a \times 3^b \times 5^c$, and numbers of this form are called **regular**. This guarantees that the quantities $\delta_n = d_n/l_n$ and their squares δ_n^2 can be written as finite sequences of sexagesimal digits without approximation. In other words, all the values in P322 are exact – and this is important because it distinguishes P322 from all modern day trigonometric tables.

While the entries b_n and d_n appear not to have any pattern as we read down the columns, the right triangles corresponding to the Pythagorean triples turn out to have a steadily decreasing reciprocal slope, or *ukullû*

$$\beta_n = \frac{b_n}{l_n}$$

from 59.30 to 37.20. In terms of angles, which would be foreign to OB thinking, the range of inclination goes from 45° to 59°, with each row separated by about 1° (Neugebauer and Sachs, 1945, 39). This observation alone suggests a trigonometric interpretation of the tablet.

Buck (1980, 340) offers another surprising and significant number-theoretical observation: eight of the fifteen entries of III' are *prime numbers*. These are, along with their decimal representations in brackets, the numbers: $d_4\,(=18\,541)$, $d_5\,(=97)$, $d_7\,(=3541)$, $d_8\,(=1249)$, $d_9\,(=769)$, $d_{10}\,(=8161)$, $d_{14}\,(=3229)$ and $d_{15}\,(=53)$. In addition we point out that eleven of the entries in column II' are irregular numbers larger than 60, and in particular the row 9 entry $b_9\,(=541)$ is prime.

The last column IV' simply contains the word *ki,* meaning "its place", together with the row number n.

In OB terminology, both right triangles and rectangles have a 'short side' (*sag*) and 'long side' (*uš*). The word for the 'hypotenuse' of a right triangle or the 'diagonal' of rectangle is *siliptum* (Neugebauer and Sachs, 1945, 52), so it is difficult to determine if the text is about rectangles or right triangles. Friberg (2007, 449) favors the rectangle interpretation, but we are agnostic on this issue. We use the notation (b, l, d) for a Pythagorean triple of numbers satisfying $b^2 + l^2 = d^2$, and these can be interpreted equally as the measurements of a rectangle or right triangle, or both simultaneously.

We are not the first to propose that the missing fragment of P322 contained the ratios $\beta_n = b_n/l_n$ and $\delta_n = d_n/l_n$. But we do make the novel proposition that the extant contents of P322 should be viewed as a description of the third ratio β_n/δ_n or δ_n/β_n required for a trigonometric table. Due to the OB preference for precision, this ratio has been replaced by three columns so that it can be presented without approximation. The squared value in column I' is an index, and columns II' and III' contain b_n and d_n, which are the simplified values β_n and δ_n with regular common factor $1/l_n$ removed, to help the scribe approximate $b_n/d_n = \beta_n/\delta_n$ or $d_n/b_n = \delta_n/\beta_n$.

3. Ratio-based trigonometry in the ancient world

Geometry in ancient Babylon arose from the practical needs of administrators, surveyors, and builders (Britton et al., 2011, 547). From their measurements of fields, walls, poles, buildings, gardens, canals, and ziggurats, a metrical understanding of the fundamental types of practical shapes was forged; typically *squares, rectangles, trapezoids* and *right triangles* (Figure 2), with general triangles very much of secondary interest. We call the OB ratio-based framework for the study of triangles "Babylonian Exact Sexagesimal Trigonometry" to distinguish it from the more familiar modern angle based framework.

Ratio-based measurements are also found in ancient Egypt, where the term *seqed,* or *sqd,* refers to the reciprocal of the slope of an inclined side in Egyptian architecture (Imhausen, 2003, 263). This was a

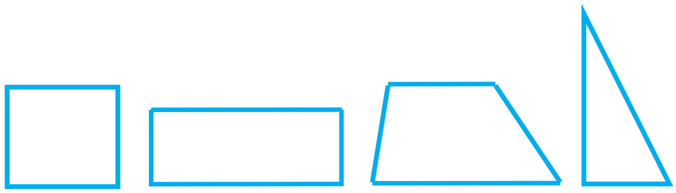

Figure 2. The key shapes in OB geometry.

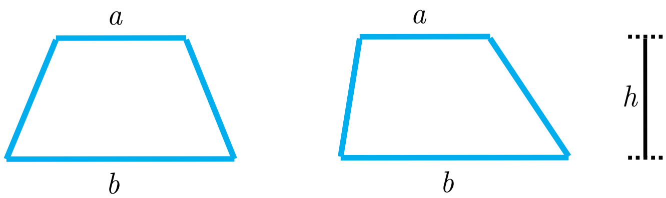

Figure 3. Different slopes, but the same *indanum*.

prominent measurement used to describe pyramids. According to Gillings (1982, 212):

> The seked of a right pyramid is the inclination of any one of the four triangular faces to the horizontal plane of its base, and is measured as so many horizontal units per one vertical unit rise. It is thus a measure equivalent to our modern cotangent of the angle of slope. In general, the seked of a pyramid is a kind of fraction, given as so many palms horizontally for each cubit vertically, where 7 palms equals one cubit.

Robins and Shute (1985, 113) state:

> We know from the pyramid exercises in the Rhind mathematical papyrus ... that the ancient Egyptians used a simple trigonometry for determining architectural inclinations...

A similar notion to *seqed* figures in OB geometry. Thureau-Dangin (1938, xvii) observed that the Akkadian *ukullû* ("fruit") was used for the reciprocal slope, or run over rise, and presents half a dozen instances of its occurrence, usually in terms of a certain length per cubit. A similar concept applied to a trapezoid appears in the "little canal problems" from YBC 4666, where we find:

> 21. A little canal. 5 UŠ is <the length>, 3 kùš the upper width, 2 kùš the lower width, 2 kùš its depth, 10 gín (volume) the assignment. What is the inclination per 1 kùš depth? $\frac{1}{2}$ kùš is the inclination.
> 22. A little canal. 5 UŠ is <the length>, 2 kùš the lower width, 2 kùš its depth, [10] gín (volume) the assignment; the inclination per 1 kùš depth is $\frac{1}{2}$kùš. [What is] the upper width? (Neugebauer and Sachs, 1945)

The same concept appears several times in MS 3052 # 1 (Friberg, 2007, 258) where the change in the width per unit height is called *indanum*. A trapezoid with height h and parallel sides of length a and b, where $a < b$, thus has an *indanum* of

$$\frac{b-a}{h}.$$

This appears strange to our modern sensibility; it indirectly combines the *ukullû* (or slopes) of two slanting sides, and so is a *feature of the entire shape* rather than of one of its sides. The two trapezoids in Figure 3, for example, have the same *indanum*.

Table 2

n	$\overline{n}$	n	$\overline{n}$	n	$\overline{n}$
2	30	16	3.45	45	1.20
3	20	18	3.20	48	1.15
4	15	20	3	50	1.12
5	12	24	2.30	54	1.06.40
6	10	25	2.24	1	1
8	7.30	27	2.13.20	1.04	56.15
9	6.40	30	2	1.12	50
10	6	32	1.52.30	1.15	48
12	5	36	1.40	1.20	45
15	4	40	1.30	1.21	44.26.40

In the case of a right triangle, the notions of *indanum* and *ukullû* overlap, and was also used to measure the steepness of walls (Robson, 1999, 90) and a grain pile (Robson, 1999, 222). Hence both Egyptian and OB cultures had a practical ratio-based measurement which is the reciprocal of our notion of slope—at least 1000 years before angles were introduced.

OB scribes also knew that the sides of *similar triangles* are in the same ratio. For example in YBC 8633 #1 the $(3, 4, 5)$ triangle is enlarged to a triangle with measurements $(1.0, 1.20, 1.40)$ through multiplication by a scaling factor called the *makşarum*, which translates as "to bind, tie", determined by the ratio of the lengths $\frac{1.20}{4} = 20$ (Neugebauer and Sachs, 1945, 53).

3.1. The Diagonal rule

Evidence of the *Diagonal rule*, which now we call Pythagoras' theorem, can be found in quite a few tablets (Høyrup, 1999, 396–401; Melville, 2004, 150–152; Friberg, 2007, 450). This demonstrates the crucial metrical understanding that the three sides of a right triangle, or the sides and diagonal of a rectangle, (b, l, d) are related by the equation

$$b^2 + l^2 = d^2. \tag{1}$$

Right triangles and rectangles of unit height $l = 1$ appear to have been particularly significant, and we call a right triangle (or rectangle) with sides $(\beta, 1, \delta)$ **normalized** when β and δ are sexagesimal numbers. These quantities notably capture the *ratios* involved in a corresponding un-normalized similar right triangle. For a normalized right triangle this is naturally connected to the idea of *ukullû*, while for a normalized rectangle β is a "statement of the relative flatness of the rectangles" (Britton et al., 2011, 539).

4. OB numerical algorithms

This section focuses on OB arithmetic and numerical procedures, particularly the methods of finding square roots, or approximate square roots, which is central to our argument on account of its direct connection with the Diagonal rule. We first explicitly lay out the available computational tools, followed by the mathematical limitations on OB square root calculations.

4.1. Reciprocal tables

Regular numbers had a special place in OB arithmetic and were used for division. Specifically, division by the regular number n (the *igi*) was performed by looking up its reciprocal $\overline{n}$ (the *igibi*) on a table of reciprocals, followed by another look up on the multiplication table for $\overline{n}$. The 'standard' table (see Table 2) lists 30 reciprocal pairs (Neugebauer and Sachs, 1945, 11), but there are variations which omit the reciprocal pairs $(1.12, 50)$, $(1.15, 48)$, $(1.20, 45)$ or add $(\frac{2}{3}, 40)$ at the beginning (Friberg, 2007, 68).

Table 3

n	$\overline{n}$
1.40	36
3.20	18
6.40	9
13.20	4.30
26.40	2.15
53.20	1.07.30
1.46.40	33.45
3.33.20	16.52.30
7.06.40	8.26.15
14.13.20	4.13.07.30

Reciprocals of irregular numbers, such as $\frac{1}{7}$, do not exist in this system. Instead, division by an irregular number would be performed by multiplication by an approximate reciprocal, such as those found in YBC 10529 (Neugebauer and Sachs, 1945, 16). Nevertheless, this sexagesimal system allows for more division calculations to be performed exactly compared with our decimal system.

The tablet CBS 29.13.21 is another reciprocal table, although of a different kind (Neugebauer and Sachs, 1945, 13). On the obverse, the tablet lists 30 reciprocal pairs. The initial entry is $(2.05, 28.48)$, while the next entry is $(4.10, 14.24)$, and was computed by respectively doubling and halving the values of the previous entry:

$$(2 \times 2.05, \overline{2} \times 28.48) = (4.10, 14.24).$$

This procedure is repeated a further 28 times, ending with the reciprocal pair

$$(1.26.18.9.11.6.41, 41.42.49.22.21.12.39.22.30).$$

The reverse contains more reciprocal pairs generated in the same manner, but from different initial reciprocal pairs. For our purpose the most relevant entries are those generated from $(1.40, 36)$, see Table 3.

4.2. The square side rule

The OB square root table enumerated the first 59 integer square roots $\text{sq.rt.}(1) = 1$, $\text{sq.rt.}(4) = 2$ up to $\text{sq.rt.}(58.01) = 59$, and was a common piece of scribal equipment. Extended square root tables are less common, such as Ashmolean 1923-366 (cdli no. P254958) which is broken but contains an additional eight square roots up to $\text{sq.rt.}(1.14.49) = 1.07$ (Neugebauer and Sachs, 1945, 34).

In certain circumstances, scribes could find the square root of larger perfect squares, for example $\text{sq.rt.}(1.44.01) = 1.19$ from IM 54 472 (Friberg, 2000, 110) or $\text{sq.rt.}(28.36.06.06.49) = 5.20.53$ from Si. 428 (Neugebauer, 1935, 80). The procedure used in these examples is well understood; Friberg (2007, 305) calls it the *square side rule* but it is perhaps more familiar as *Heron's method*.

The technique may be described in modern algebraic terms as follows: to find $\text{sq.rt.}(a)$, first choose a regular number n such that n^2 is approximately a, written $n^2 \simeq a$. Then calculate

$$\text{sq.rt.}(a) \simeq \overline{2}\,(a\overline{n} + n)\,.$$

In the example from IM 54 472 we see that $n = 1.20$ is a regular number and $(1.20)^2 = 1.46.40 \simeq 1.44.01$, so $\text{sq.rt.}(1.44.01)$ is approximately

$$\overline{2}\left(1.44.01 \times \overline{1.20} + 1.20\right) = 1.19$$

which is in fact the square root exactly. Indeed, the question was carefully crafted, and the number 1.44.01 "deliberately chosen in such a way that it would be easy to find the square root" by the square side rule (Friberg, 2000, 111). Similarly in the example from Si. 428, $n = 5.20.00$ is a regular number such that $(5.20.00)^2 = 28.26.40.00.00 \simeq 28.36.06.06.49$, so sq.rt.(28.36.06.06.49) is approximately

$$\overline{2}\left(28.36.06.06.49 \times \overline{5.20.00} + 5.20.00\right) = 5.20.53; 04.23.20.37.30 \simeq 5.20.53.$$

This raises the question: how did the scribe find the initial regular approximation 5.20.00? In the cases above, the regular approximations are $1.46.40 = 2^8 \times 5^2$ and $28.26.40.00.00 = 2^{12} \times 5^2 \times (1.00)^2$. The relatively high powers of 2 involved suggest that an extended reciprocal table similar to CBS 29.13.21 was used as a look-up table of regular squares. Indeed, 1.46.40 is actually on CBS 29.13.21, and 28.26.40 is the next entry if the pattern were to be continued for one more iteration. In addition to regular squares n^2, CBS 29.13.21 also contains the numbers n and $\overline{n}$ used in the approximation itself, after multiplication by $\overline{5}$ and 5: $(\overline{5} \times 6.40, 5 \times 9) = (1.20, 45)$ and $(\overline{5} \times 26.40, 5 \times 2.15) = (5.20, 11.15)$. So it seems likely that the square side rule utilized an auxiliary extended reciprocal table to look up an initial approximation and reciprocal. In any case, the success of the square side rule depends upon the accuracy of the initial approximation – however it was obtained by the scribe.

As numbers become large the regular numbers become increasingly sparse, and the square side rule ceases to produce good results. For example, using a modern computer we find that the best initial regular approximation to sq.rt.(26.31.31.18.01) is $5.03.45 = 3^6 5^2$. But even with this optimal approximation the square side rule gives

$$\overline{2}\left(26.31.31.18.01 \times \overline{(5.03.45)} + (5.03.45)\right) = 5.09.03; 44.22.19.15.33.20 \simeq 5.09.04.$$

Since the correct value is 5.09.01 we see that this is beyond what can be achieved by the square side rule alone.

In addition to the theoretical limitations, the practical problem of finding an initial regular approximation is especially pertinent. Problem xviii from the combined tablet fragments BM 96957 + VAT 6598 ask the scribe to approximate the diagonal d in the rectangle where the sides $b = 10$ and $l = 40$ are known, and the approximation is obtained by the square side rule. Because the rectangle is long and flat the long side is close to the diagonal, and since 40 is regular, it can be used as the initial regular approximation to the diagonal to calculate:

$$d = \text{sq.rt.}(b^2 + l^2) \simeq \overline{2}\left((b^2 + l^2)\overline{l} + l\right) = b^2(\overline{2l}) + l = 41; 15.$$

The same procedure can be used in problem xix, which is to find the short side of the rectangle with length 40 and diagonal 41; 15. But here the initial approximation is not given in the question, and the scribe chooses instead to apply the square side procedure backwards to re-obtain the original short side of the rectangle. The most interesting case is in problem xx, where the scribe is asked to recover the long side of the rectangle with short side 10 and diagonal 41; 15. Again the square side rule could have been used but the scribe declines, and says there is no solution.

This unusual approach to the second and third problems highlights the non-trivial nature of the square side rule and the general difficulty with calculating square roots.

4.3. The square root and trailing part algorithms

An important observation is that small regular factors of a sexagesimal number are apparent from the digits of that number: if the least significant digit is a multiple of two, three or five, then the number itself is a multiple of two, three or five. The next two OB numerical algorithms make use of this property.

Table 4

Number	Square factor	Action
1.7.44.3.45	3.45	× 16
18.3.45	3.45	× 16
4.49	4.49	terminate.

Table 5

Number	Square factor	Action
1.53.10.29.32.52.16	16	× 3.45
7.04.24.20.48.16	16	× 3.45
26.31.31.18.01	26.31.31.18.01	terminate.

Table 6

Number	Square factor	Action
35.10.02.28.27.24.26.40	6.40	× 9
2.11.52.39.16.42.46.40	6.40	× 9
8.14.32.27.17.40.25	25	× 2.24
1.53.56.32.01	1.53.56.32.01	terminate.

The *square root algorithm* allowed a scribe to extend the reach of the square root table or square side rule. This algorithm was an iterative procedure which successively identified and removed *square* regular factors to reduce the problem to the case where sq.rt.(a) can be solved by the methods described above. In UET 6/2 222 for example, it can be seen from the least significant digits of the number 1.7.44.03.45 that it contains the square regular factor 3.45 $(= 15^2)$ (Proust, 2012, 21). The scribe records and then removes this factor from the number through multiplication by $\overline{3.45} = 16$. The process continues until the scribe is left with 4.49, which can be identified as 17^2 using the standard square root table, see Table 4.

The square root is then given as the product of the square roots of each square factor:

$$\begin{aligned}\text{sq.rt.}(1.7.44.3.45) &= \text{sq.rt.}(3.45) \times \text{sq.rt.}(3.45) \times \text{sq.rt.}(4.49)\\ &= 15 \times 15 \times 17 = 1.3.45.\end{aligned}$$

For our argument, the most important feature of this algorithm is that problems involving the Diagonal rule, such as MS 3971 # 3, were carefully constructed to ensure that square roots could be easily found using this method.

To exemplify the difficulty of the general case, consider the challenging task of computing the square root of the column I' row 4 entry δ_4^2 of P322. The steps of the square root algorithm are described in Table 5.

With modern computational assistance we may find that sq.rt.(26.31.31.18.01) = 5.09.01 but this would have been beyond reach of an OB scribe using standard techniques, because the number lacks a regular square approximation which is close enough to make the square side rule accurate, and further simplification of the question is impossible because $d_4 = 5.09.01\ (= 18,541)$ is prime. Current theories regarding the purpose of P322 posit that a scribal student can perform the square root procedure, but these theories are based on examples that are computationally trivial. There is no evidence to suggest that a scribal student was ever required to calculate a square root as complex as sq.rt.(δ_4^2).

Another example is the problem of finding sq.rt.(β_{10}^2). The steps of the square root algorithm are described in Table 6.

The task is then to compute sq.rt.(1.53.56.32.01) by the square side rule. Using the optimal approximation $n = 2^3 \times 5^4 = 1.23.20$ the scribe can apply the square side rule to obtain 1.22.41; 09.07.33.36 $\simeq$

Table 7

First number	Second number	Common regular factor	Action
58.27.17.30	1.23.46.02.30	30	$\times$ 2
1.56.54.35	2.47.32.05	5	$\times$ 12
23.22.55	33.30.25	5	$\times$ 12
4.40.35	6.42.05	5	$\times$ 12
56.07	1.20.25	none	terminate.

$1.22.41 = b_{10}$. But this level of accuracy comes at a price: the scribe must have access to exotic tables of regular squares such as $n^2 = 2^6 \times 5^8 = 1.55.44.26.40$. Otherwise, the scribe must choose a sub-optimal initial approximation and the square side rule produces an inaccurate approximation.

In summary, finding the square root of either the β^2 or δ^2 values implicit in column I' is unusually difficult for an OB scribe, if not impossible.

The second numerical algorithm which is important for an understanding of P322 is the *trailing part algorithm* (Friberg, 2007, 37). This is an iterative process for identifying regular factors of a number from the least digits, or trailing parts, and then removing these factors through multiplication by the corresponding reciprocals. The method can be used to remove the common regular factors from two numbers. For example, the least significant digits of the numbers $\beta_2 = 58.27.17.30$ and $\delta_2 = 1.23.46.02.\underline{30}$ share a common regular factor of 30, and this can be removed through multiplying both numbers by $\overline{30} = 2$. This process can be repeated until there are no remaining common regular factors, see Table 7.

5. Reconstructing P322

There are two main theories as to how an OB scribe might have generated P322. The original proposal of Neugebauer and Sachs (1945, 40), modified by de Solla Price (1964), and more recently by Proust (2011, 663), emphasizes the role of two generators r and s used to create the Pythagorean triple $\left(\bar{2}(r\bar{s} - s\bar{r}), 1, \bar{2}(r\bar{s} + s\bar{r})\right)$, while Bruins' theory (1949, 1957), supported by Robson (2001, 194), claims that a reciprocal pair $(x, \bar{x})$ was used to create normalized Pythagorean triples as $\left(\bar{2}(x - \bar{x}), \text{sq.rt.}(x\bar{x}), \bar{2}(x + \bar{x})\right)$. The relative merits of both points of view, particularly with respect to the errors on the tablet, are well presented by Britton et al. (2011).

We propose a modification of these already established theories which blends their respective advantages. Expanding upon the work of Proust (2011, 664), we give an explicit procedure by which the scribe first iterates through the standard table of reciprocals for the values of s, and then finds all possible corresponding values of r. Furthermore, we have a different suggestion for why the procedure terminated. Here are the steps that we believe the scribe followed in order to create P322.

1. Let s run through all the entries of a standard reciprocal table, starting with the first reciprocal pair $(s, \bar{s}) = (2, 30)$.
2. For each such value of s, consult the multiplication table of $\bar{s}$ to find $r\bar{s}$, where r satisfies both
$$s < r \qquad \text{and} \qquad r\bar{s} < 2.24.$$
3. For each such r, look up $\bar{r}$ in the standard reciprocal table and find $s\bar{r}$ on the appropriate multiplication table.
4. Remove any duplication in the list of $(r\bar{s}, \bar{r}s)$ values.
5. Calculate the values
$$\beta = \bar{2}(r\bar{s} - s\bar{r}) \qquad \text{and} \qquad \delta = \bar{2}(r\bar{s} + s\bar{r}).$$

Table 8

s	r	s	r	s	r
2	3	15	16, 32	36	
3	4, 5	16	25, 27	40	1.21
4	5, 9	18	25	45	1.04
5	6, 8, 9, 12	20	27	48	
6		24	25	50	1.21
8	9, 15	25	27, 32, 36, 48, 54	54	2.05
9	10, 16, 20	27	32, 40, 50, 1.04	1	2
10		30			
12	25	32	45, 1.15		

6. Repeat step 1 with the next reciprocal pair until you see $(\beta, \delta) = (45, 1.15)$.
7. Sort the values (β, δ) into decreasing order (38 terms).
8. Calculate δ^2, or equivalently $\beta^2 + 1$, for each pair (β, δ).
9. For each pair (β, δ), perform the trailing part algorithm to find b and d, along the way producing the multiplier l.
10. Perform a check that indeed $b^2 + l^2 = d^2$.
11. Record the values $\beta, \delta, \delta^2, b, d$ and row number.

In step 2, the restriction $r\bar{s} < 2; 24$ is equivalent to the OB convention that $\beta = \bar{2}(r\bar{s} - s\bar{r})$ is always the short side (de Solla Price, 1964, 223).

In step 4, this could equivalently have been done by taking unique pairs in step 7.

In step 6, we suggest that the process terminated with $s = 1, r = 2$, which yields the canonical (0; 45, 1, 1; 15) triangle. This may have been the original intention of the scribe. Note that the 37-th iteration is the only one which requires a non-standard reciprocal $(r, \bar{r}) = (2.05, 28.48)$, so perhaps the scribe was unwilling to generate further non-standard reciprocals. This may serve to explain the observation of de Solla Price (1964, 222) that s is bounded by 1.00.

In step 8, we propose that the scribe computed $\delta^2 = (\bar{2}(r\bar{s} + s\bar{r}))^2$ as an aside. Unlike the method used in numerically simpler tablets such as MS 3052 # 2 (Friberg, 2007, 275), the computational difficulty in calculating sq.rt.(δ^2) or sq.rt.(β^2) makes it more likely that β and δ were calculated directly from the reciprocals. If β^2 or δ^2 were used to first calculate β or δ and subsequently simplified to b or d in step 9 then we should expect to see computational errors in column I' carried over to columns II' and III'. As this is not the case we have another reason for believing that β and δ were not calculated from β^2 and δ^2.

In step 9, we agree with Bruins (1949, 1957) that the trailing part algorithm was applied to β and δ to obtain the regular factor $l = 2rs$ and the simplified values $b = r^2 - s^2$ and $d = r^2 + s^2$. A table which shows explicitly how the trailing part algorithm was applied, at least to the 15 rows of P322, is given by Britton et al. (2011, 536), which also discusses how the trailing part algorithm explanation gives plausible reasons for most of the errors.

Step 10 is suggested by the error in row 13, where b_{13}^2 is written instead of b_{13}.

Step 11 contains the conjectured values of the completed table, as we discuss in the next section.

Table 8 exhibits the values of s and r so obtained from the 38 iterations of this generation procedure, which except for the first and last entries also appears in Britton et al. (2011, 533).

As observed by de Solla Price (1964, 5), when arranged in descending order according to δ^2, this method produces the first 15 rows of P322. The remaining entries would have exactly filled the blank rows on the bottom and reverse of the tablet, and so is likely to be the content of the unfilled rows.

This is an improvement on Bruins' theory, which depended upon the scribe having access to a very specific non-standard reciprocal table containing exactly the entries $(r\bar{s}, s\bar{r})$. For the first 15 rows of P322 the closest known match is AO 6456 from many centuries later in the Seleucid era (Bruins, 1957, 631). However, there is merit in this theory because of its procedural aspect, which we have used in our constructive approach, but with only a standard table of reciprocals along with multiplication tables. This is a small improvement to the recent theory of Proust (2011).

5.1. What did P322 originally look like?

To introduce our main argument that P322 is an exact sexagesimal trigonometric table text, it is helpful to consider the current best thinking about what the table was originally intended to look like. The important suggestion that β and δ should be the contents of the missing columns was first made by Friberg (1981, 295). This view is also supported by Britton et al. (2011, 539), citing consistency with the column headings, method of generation, and noting that the space required for these values matches the projected original dimensions. This fits with our trigonometric interpretation as well, so we are in agreement that the missing columns I and II probably contained β and δ, and that the full tablet likely contained six columns I, II, I', II', III' and IV' consisting of the values $\beta_n, \delta_n, \delta_n^2, b_n, d_n$ and the row number n. The value of β_n here is significant as a measurement of steepness, either as the *ukullû* or *indanum* of a right triangle or flatness of a rectangle.

We also agree with the many authors (de Solla Price, 1964; Conway and Guy, 1996; Britton et al., 2011) who have argued that the top, bottom and reverse of the tablet were intended to be filled with an additional 23 rows that can be generated in the systematic fashion originally suggested by de Solla Price (1964). We refer to the full table with all 38 reconstructed rows and six columns as P322(CR), and it is shown here in full except for the original column headings (see Table 9).

It is clear that P322 is a remarkable outlier among OB mathematics tablets in terms of the amount of work required to create it. We should not then be surprised if the purpose of this tablet is also unlike that of any other tablet that we have uncovered so far.

5.2. Current theories about the purpose of P322

Despite the huge amount of study and scholarship that has been invested in understanding the errors and method of generation of P322, relatively little progress has been made as to the tablet's intended purpose. We will of course argue that P322 is an exact sexagesimal trigonometric table text. But first we examine and critique other current views as to why the OB scribe created the tablet.

One argument, which is presented by many authors (Buck, 1980; Robson, 2002; Friberg, 2007, 433), is that P322 served as a teacher's aide relating to quadratic *igi-igibi* problems. It is to be noted however that Friberg (1981) previously championed the idea that P322 was a classification of certain types of normalized right triangles.

Quadratic problems featuring reciprocal pairs $(x, \overline{x})$ involve two closely related OB geometric constructions: *completing the square* and *generating a normalized right triangle*. The former concerns a rectangle with side lengths x and $\overline{x}$; the latter concerns a right triangle with side lengths 1 and $\bar{2}(x - \overline{x})$, and with diagonal $\bar{2}(x + \overline{x})$.

Completing the square style problems, known as *igi-igibi* problems, ask a student to find a reciprocal pair x and $\overline{x}$ whose sum or difference is known. For example in YBC 6967 a student is asked to solve $x - \overline{x} = 7$. We have inserted the "radix" point ";" in the calculations below, as this interpretation is clear from the arithmetic. The student follows the standard procedure, which can be described as follows: transform the rectangle of sides x and $\overline{x}$ into a square, by first rearranging it into a gnomon or L-shaped region. Add a square with side length $\bar{2}(x - \overline{x}) = \bar{2} \times 7 = 3;30$ to obtain a larger square with side length $\bar{2}(x + \overline{x})$ (Høyrup, 2014, 9). Algebraically, this is

$$\left(\bar{2}(x - \overline{x})\right)^2 + x\overline{x} = (3;30)^2 + 1.00; = 12;15 + 1.00; = 1.12;15 = \left(\bar{2}(x + \overline{x})\right)^2.$$

Table 9

$I:\beta$ base	$II:\delta$ diagonal	$I':\delta^2$ diagonal squared	$II':b$	$III':d$	IV'
59.30	1.24.30	1.59.00.15	1.59	2.49	1
58.27.17.30	1.23.46.02.30	1.56.56.58.14.50.06.15	56.07	1.20.25	2
57.30.45	1.23.06.45	1.55.07.41.15.33.45	1.16.41	1.50.49	3
56.29.04	1.22.24.16	1.53.10.29.32.52.16	3.31.49	5.09.01	4
54.10	1.20.50	1.48.54.01.40	1.05	1.37	5
53.10	1.20.10	1.47.06.41.40	5.19	8.01	6
50.54.40	1.18.41.20	1.43.11.56.28.26.40	38.11	59.01	7
49.56.15	1.18.03.45	1.41.33.45.14.03.45	13.19	20.49	8
48.06	1.16.54	1.38.33.36.36	8.01	12.49	9
45.56.06.40	1.15.33.53.20	1.35.10.02.28.27.24.26.40	1.22.41	2.16.01	10
45	1.15	1.33.45	45	1.15	11
41.58.30	1.13.13.30	1.29.21.54.02.15	27.59	48.49	12
40.15	1.12.15	1.27.00.03.45	2.41	4.49	13
39.21.20	1.11.45.20	1.25.48.51.35.06.40	29.31	53.49	14
37.20	1.10.40	1.23.13.46.40	28	53	15
36.27.30	1.10.12.30	1.22.09.12.36.15	2.55	5.37	16
32.50.50	1.08.24.10	1.17.58.56.24.01.40	7.53	16.25	17
32	1.08	1.17.04	8	17	18
30.04.53.20	1.07.07.06.40	1.15.04.53.43.54.04.26.40	1.07.41	2.31.01	19
29.15	1.06.45	1.14.15.33.45	39	1.29	20
27.40.30	1.06.04.30	1.12.45.54.20.15	6.09	14.41	21
25	1.05	1.10.25	5	13	22
24.11.40	1.04.41.40	1.09.45.22.16.06.40	14.31	38.49	23
22.22	1.04.02	1.08.20.16.04	11.11	32.01	24
21.34.22.30	1.03.45.37.30	1.07.45.23.26.38.26.15	34.31	1.42.01	25
20.51.15	1.03.31.15	1.07.14.53.46.33.45	16.41	50.49	26
20.04	1.03.16	1.06.42.40.16	5.01	15.49	27
18.16.40	1.02.43.20	1.05.34.04.37.46.40	5.29	18.49	28
17.30	1.02.30	1.05.06.15	7	25	29
14.57.45	1.01.50.15	1.03.43.52.35.03.45	6.39	27.29	30
13.30	1.01.30	1.03.02.15	9	41	31
11	1.01	1.02.01	11	1.01	32
10.14.35	1.00.52.05	1.01.44.55.12.40.25	4.55	29.13	33
7.05	1.00.25	1.00.50.10.25	17	2.25	34
6.20	1.00.20	1.00.40.06.40	19	3.01	35
4.37.20	1.00.10.40	1.00.21.21.53.46.40	52	11.17	36
3.52.30	1.00.07.30	1.00.15.00.56.15	31	8.01	37
2.27	1.00.03	1.00.06.00.09	49	20.01	38

Table 10

Number	Square factor	Action
1.12; 15	0; 15	$\times$ 4
4.49	4.49	terminate.

The square root algorithm is then employed to calculate the half sum $\bar{2}(x+\bar{x}) = \text{sq.rt.}(1.12;15)$, see Table 10.

So $\bar{2}(x+\bar{x}) = \text{sq.rt.}(1.12;15) = \text{sq.rt.}(4.49) \times \text{sq.rt.}(0;15) = 17 \times 0;30 = 8;30$. With the half sum and half difference in hand, the scribe concludes that $x = 8;30 + 3;30 = 12$ and $\bar{x} = 8;30 - 3;30 = 5$. We see that $x\bar{x} = 1$ was taken as 1.00; by the addition $12;15 + x\bar{x} = 1.12;15$, and the sq.rt. procedure was able to keep track of the order of magnitude, as is indicated by the addition of the half sum and half difference $x = 8;30 + 3;30 = 12$.

Table 11

Number	Square factor	Action
0; 00.15.00.56.15	0; 03.45	× 16
0; 4.00.15	0; 15	× 4
0; 16.01	0; 16.01	terminate.

Buck (1980, 344) and later Robson (2001, 200) argued that the column I' entry of P322 is an intermediate value that would allow a teacher to check student working for an *igi-igibi* geometric construction such as YBC 6967; P322 could thus be used to check that a student correctly computed $(\bar{2}(x-\bar{x}))^2$ or $(\bar{2}(x+\bar{x}))^2$ before using the sq.rt. procedure to calculate the half difference $\bar{2}(x-\bar{x})$ or half sum $\bar{2}(x+\bar{x})$.

There is a fundamental mathematical difficulty in such an explanation. The key operation upon which all quadratic problems hinge is the extraction of a square root. But as we suggested in our discussion of square root algorithms, the numbers on P322 are just too big to allow students to reasonably obtain the square roots of the quantities required.

In contrast, the kinds of *igi-igibi* problems that actually appear on known tablets have square root calculations which can be performed using a square root table instead of the square side rule, and are vastly simpler than those implicit in P322.

Furthermore, if P322 is a table of parameters for quadratic problems, then this does not explain the comprehensive and systematic ordering by *ukullû*. In addition, why would such a table be ordered by the answers and not the questions? Nor does this hypothesis explain why, in the generation procedure, the reciprocal pairs $(r\overline{s}, \overline{r}s)$ were interpreted as multiplying to 1, which is an unnecessary restriction for *igi-igibi* problems. This hypothesis also fails to explain the purpose of columns II' and III'.

A reciprocal pair $(x, \bar{x})$ can be used to exhibit the geometry of a normalized right triangle with sides $(\beta, 1, \delta)$, as

$$(\beta, 1, \delta) = \left(\bar{2}(x-\bar{x}), \text{sq.rt.}(x\overline{x}), \bar{2}(x+\bar{x})\right). \quad (2)$$

It was observed by Britton et al. (2011) that the construction forces the scribe to interpret the reciprocals such that $x\overline{x} = 1$. This condition, together with the different geometric emphasis, distinguishes the normalized right triangle problems from the *igi-igibi* problems. Algebraically, however, the procedures have a lot in common. For example, MS 3052 #2 and MS 3971 # 3 use the *igi* and *igibi* to compute the diagonal δ as in (2), but then use the Diagonal rule to compute $\beta = \text{sq.rt.}(\delta^2 - 1)$.

Take the first problem from Friberg's reconstruction of MS 3971 # 3e for example. We see an *igi* $x = 1; 04$ and *igibi* $\overline{x} = 0; 56.15$. Following the procedure, we compute the diagonal $\delta = 1; 00.07.30$, the squared diagonal $\delta^2 = 1; 00.15.00.56.15$ and then use the square root algorithm on $\delta^2 - 1$, see Table 11.

Since $\text{sq.rt.}(16.01) = 31$ is on the standard square root table, the scribe concludes that

$$\beta = \text{sq.rt.}(0; 03.45) \times \text{sq.rt.}(0; 15) \times \text{sq.rt.}(0; 16.01) = 0; 15 \times 0; 30 \times 0; 31 = 0; 03.52.30.$$

Here we have assumed that the scribe is able to keep track of the sexagesimal place, as this seems to be the case as indicated by YBC 6967, and it ensures that $\text{sq.rt.}(0; 15) = 0; 30$ as opposed to the irrational number $\text{sq.rt.}(15)$.

The author must have gone to considerable effort to find suitable *igi* and *igibi* which ensure easy computation of the square root; precisely the opposite can be said for the author of P322 who seems untroubled by the difficulty of square root computations.

The rectangle in MS 3052 #2 corresponds to row 11 of P322, and the five rectangles of MS 3971 #3 correspond respectively to rows 37, 18, 22, 29 and 32 of P322(CR). Friberg (2007, 436) claims this is proof of a connection between these problems and P322, but this is an overstatement. On the contrary these examples confirm only the scribal school tradition of creating problems that can be solved using the square root table. It is clear that P322 goes well beyond this tradition: of the 38 rectangles in P322(CR) only 13 have short sides that can be computed in this way.

Friberg (2007, 433) argues that P322 is a table of parameters for computing the square root in texts such as MS 3971 # 3. Following the square root algorithm at first, square regular factors are iteratively removed from β^2 or δ^2 until only the irregular *factor reduced core* remains, and the value b or d is then the required value for the final step of the sq.rt. algorithm. This is a nice hypothesis, but it is not consistent with the numbers found in columns II' and III'. Specifically, entries b_5, b_{11}, d_{11} and b_{15} contain regular factors and hence are not factor reduced cores. Furthermore, rows 11 and 15 do not belong in a table of factor reduced cores: row 11 does not belong because b_{11} and d_{11} are entirely regular, and row 15 does not belong because the square root can be found using a standard square root table.

Britton et al. (2011, 559) suggest that columns II' and III' might have been the solutions to some challenging scribal problem involving a normalized rectangle with diagonal δ and width β, and a simplified rectangle with diagonal d and width b. Not only is the level of difficulty here outside the scribal tradition, but as discussed above β^2 and δ^2 can not be used to find the simplified values of b and d in every case. Furthermore, why would a table of harder problems also contain the trivial case in row 11?

Friberg (1981, 300) originally suggested that P 322 might be a classification of right triangles:

> it may have been the intention of the author of the tablet to find the front and diagonal of all rational right triangles with flank (i.e., length) = 1 under the sole condition for practical reasons that the parameter $t = s/r$ ... must be a regular sexagesimal number such that, for instance, $s < 1.0$.

Robson (2001, 201) dismisses this possibility because it fails to explain the purpose of column I', and questions the absence of l alongside the columns for width and diagonal. However there is merit in Friberg's claim that the tablet is a comprehensive description of right triangles. Indeed this claim was recently repeated by Britton et al. (2011, 561) in the context of rectangles. We feel that this observation is in the right direction, and we will later demonstrate the role of column I' as an index.

Some authors have raised the possibility that P322 was a trigonometric table in the modern sense (Knuth, 1972, 675; Maor, 2002, 11; Buck, 1980, 344). Buck, for example, asks

> Could this tablet be a primitive trigonometric tablet, intended for engineering or astronomic use? But again, why is $\tan^2\theta$ useful?

Yet such questions lead directly to anachronism. As Robson (2002, 112) points out:

> ... there was no conceptual framework for measured angle or trigonometry [in the OB era]. In short, Plimpton 322 could not have been a trigonometric table.

But the possibility that P322 is an exact sexagesimal trigonometric table, without the assumption of a circle-based measurement system based on angles, has not been considered until now.

6. A remarkable exact sexagesimal trigonometric table

There is no known historical evidence that confirms how P322(CR) was actually used. This is a question that must be answered by archeology, or by further discoveries on existing tablets. But we can open the possibility for a trigonometric interpretation.

Table 12

Given ratio	Compute	Compare with	From column	To find the ratios
b/l or l/b	b/l	β	I	$\delta_n, b_n/d_n$ or d_n/b_n
d/l or l/d	d/l	δ	II	$\beta_n, b_n/d_n$ or d_n/b_n
d/b	$1/((d/b)^2-1)$	β^2	I'	β_n, δ_n
b/d	$1/(1-(b/d)^2)$	δ^2	I'	β_n, δ_n

We propose that each of the five columns of P322(CR) has a well defined role as an input and/or output for the essential trigonometric problem of going from one known ratio of a right triangle to finding the other two ratios. The first three columns I, II and I' of P322(CR) serve as the entry points, in the sense that any ratio of sides can be compared to one of these columns to determine a row. The approximate shape of the right triangle or rectangle can then be calculated from the entries in columns I, II, II' and III'.

For example, if you are given the ratio b/l then you can compare this value to the contents of column I to determine the closest matching row, and the ratios δ_n and b_n/d_n describe the shape. If you are given the ratio b/d then you can compute $1/(1-(b/d)^2)$ and compare this value with δ_n^2 from column I' to determine the closest matching row, and the ratios δ_n and β_n describe the shape. The full list of possibilities is given in Table 12.

Note the exact ratios β_n and δ_n serve a dual purpose both as an index and the information which determines the shape. However the ratios b_n/d_n and d_n/b_n cannot be written so concisely because of the non-regular nature of the denominators. So instead the index and information appear separately: the value δ_n^2 serves as an index when b/d is known, the value β_n^2 serves as an index when d/b is known, and the simplified values b_n and d_n are the information that allows the scribe to make their own approximation to the ratio b_n/d_n or d_n/b_n.

Once the ratios of the sides are determined, any side length can be used to scale these ratios and thereby find an approximation to the right triangle or rectangle.

Problems xviii, xix and xx from the combined tablet BM 96957 + VAT 6598, show that OB scribes were interested in trigonometric problems of this kind regarding rectangles. We have paraphrased these problems below, along with new solutions to show P322 how can be used to solve this style of problem.

Problem xviii. Suppose that you measure the short and long side of a rectangle as $b=10$ and $l=40$. What is the diagonal?

Solution. *You calculate* $\beta = b/l = 0;15$ *and then search P322(CR) for the normalized rectangle which has the closest* β_n. *In this case it is*

$$(\beta_{30}, 1, \delta_{30}) = (0;14.57.45, 1, 1;01.50.15).$$

Rescale this normalized rectangle by the makşarum $l=40$, *and read off the diagonal to get the approximation*

$$l \times \delta_{30} = 41;13.30.$$

Problem xix. Suppose that you measure $d=41.15$ and $l=40$. What is the short side of the rectangle?

Solution. *You first calculate the ratio* $\delta = d/l = 1;01.52.30$, *then search for the normalized rectangle which has the closest value of* δ_n. *This is again* $(\beta_{30}, 1, \delta_{30})$. *Rescale this normalized rectangle by the makşarum* $l=40$, *and read off the short side to get the approximation*

$$l \times \beta_{30} = 9;58.30.$$

Problem xx. Suppose that you measure $b = 10$ and $d = 41; 15$. What is the long side of the rectangle?

Solution. *First find the ratio* $d/b = 4; 07.30$. *The squared ratio is* $(d/b)^2 = 17; 00.56.15$. *Subtract* 1 *to give* $(l/b)^2 = 16; 00.56.15$. *The reciprocal is approximately* $\beta^2 \simeq 0; 3.44$. *The closest value in column* I' *is* $\beta_{30}^2 = 0; 03.43.52.35.03.45$. *The long side is approximately*

$$l \simeq \delta_n \times d - \beta_n \times b = 40; 01.10.18.45.$$

These examples establish that P322(CR) contains enough information to solve practical geometric problems of the OB era, and hence *could* have been used as an exact sexagesimal trigonometric table. But if we accept this as a possibility then the immense power of the table becomes clear. We can exhibit this power by applying it to a wider range of problems, which renders our hypothesis more likely – such a powerful trigonometric table is not created by accident.

6.1. A decimal version of P322

Table 13 is obtained by approximating the exact values in columns I, II and I' by decimal numbers to eight decimal digits: we refer to this as P322(CR-Decimal8).

The modern reader can interpret this table as follows: β and δ represent the ratios b/l and d/l of a right triangle (b, l, d), subject to the convention that $b \leq l$. The middle column I' gives $(d/l)^2$ or $(b/l)^2$ which can be compared to the ratios b/d or d/b. The quantities b and d allow us to recover the ratio b/d, or d/b. The first and most important column is β which is the reciprocal slope, or *ukullû*, of the diagonal. This varies between near 1 (which is the value for an isosceles right triangle), to near 0, which describes a triangle with a very small base relative to the height.

6.2. Plimpton 322 versus Madhava's sine table

Having made the case that P322 is a trigonometric table, we now step well outside of OB mathematics to more recent times in order to make the further claim that P322 is superior in mathematical power to much later trigonometric tables. This exact sexagesimal trigonometry should not be seen as inferior to later developments; for certain applications it may actually be superior. Certainly the possibility of having exact tables should give us a lot to ponder.

We make a comparison with another historically interesting artifact: the well-known sine table created by the Indian astronomer-mathematician Madhava (1340–1425 CE) from Kerala, fully 3000 years after P322, translated from approximate sexagesimal to 8 decimal digits (Van Brummelen, 2009, 121), see Table 14.

We do not know how Madhava created his table, but there are suggestions that he used the power series expansion for $\sin x$, which was known in Kerala at this time, very likely due to Madhava himself. The choice of angles is historically quite common: the procedure for computation of the table involved successive bisections of angles, starting with for example 30°.

We will consider two elementary and typical applications of trigonometry, each concerned with lengths in right triangles, and work to four decimal places of accuracy, with no other aids than pencil and paper.

Problem 1. *Suppose that a ramp leading to the top of a ziggurat wall is* 56 *cubits long, and the vertical height of the ziggurat is* 45 *cubits. What is the distance x from the outside base of the ramp to the point directly below the top? (See Figure 4.)*

Solution *(using P322). Use the familiar notation of a right triangle in the OB setting, and set* $l = 45$ *and* $d = 56$. *Then* $\delta = \frac{d}{l} = \frac{56}{45} \simeq 1.2444$. *The closest row on P322 is row 11 with* $\delta_{11} = 1.25$. *From columns* II'

Table 13

Base β	Diagonal δ	Squared diagonal δ^2	Simplified base b	Simplified diagonal d	Row
0.99166666	1.40833333	1.98340277	119	169	1
0.97424768	1.39612268	1.94915855	3367	4825	2
0.95854166	1.38520833	1.91880212	4601	6649	3
0.94140740	1.37340740	1.88624790	12709	18541	4
0.90277777	1.34722222	1.81500771	65	97	5
0.88611111	1.33611111	1.78519290	319	481	6
0.84851851	1.31148148	1.71998367	2291	3541	7
0.83229166	1.30104166	1.69270941	799	1249	8
0.80166666	1.28166666	1.64266944	481	769	9
0.76558641	1.25941358	1.58612256	4961	8161	10
0.75	1.25	1.5625	45	75	11
0.69958333	1.22041666	1.48941684	1679	2929	12
0.67083333	1.20416666	1.45001736	161	289	13
0.65592592	1.19592592	1.43023882	1771	3229	14
0.62222222	1.17777777	1.38716049	28	53	15
0.60763888	1.17013888	1.36922501	175	337	16
0.54745370	1.14004629	1.29970555	473	985	17
0.53333333	1.13333333	1.28444444	8	17	18
0.50135802	1.11864197	1.25135986	4061	9061	19
0.4875	1.1125	1.23765625	39	89	20
0.46125	1.10125	1.21275156	369	881	21
0.41666666	1.08333333	1.17361111	5	13	22
0.40324074	1.07824074	1.16260309	871	2329	23
0.37277777	1.06722222	1.13896327	671	1921	24
0.35954861	1.06267361	1.12927520	2071	6121	25
0.34756944	1.05868055	1.12080451	1001	3049	26
0.33444444	1.05444444	1.11185308	301	949	27
0.30462962	1.04537037	1.09279921	329	1129	28
0.29166666	1.04166666	1.08506944	7	25	29
0.249375	1.030625	1.06218789	399	1649	30
0.225	1.025	1.050625	9	41	31
0.18333333	1.01666666	1.03361111	11	61	32
0.17071759	1.01446759	1.02914449	295	1753	33
0.11805555	1.00694444	1.01393711	17	145	34
0.10555555	1.00555555	1.01114197	19	181	35
0.07703703	1.00296296	1.00593470	52	677	36
0.06458333	1.00208333	1.00417100	31	481	37
0.04083333	1.00083333	1.00166736	49	1201	38

Table 14

Angle (degs)	sin	Angle (degs)	sin
03.75	0.06540314	48.75	0.75183985
07.50	0.13052623	52.50	0.79335331
11.25	0.19509032	56.25	0.83146960
15.00	0.25881900	60.00	0.86602543
18.75	0.32143947	63.25	0.89687275
22.50	0.38268340	67.50	0.92387954
26.25	0.44228865	71.25	0.94693016
30.00	0.49999998	75.00	0.96592581
33.75	0.55557022	78.75	0.98078527
37.50	0.60876139	82.50	0.99144487
41.25	0.65934580	86.25	0.99785895
45.00	0.70710681	90.00	0.99999997

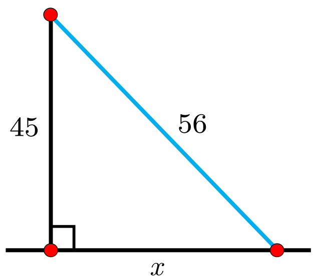

Figure 4.

and III' you find that the corresponding ratio of a regular right triangle is $\frac{b_{11}}{d_{11}} = \frac{3}{5} = 0.6$ *so that* $\frac{x}{56} \simeq 0.6$. *Thus* $x \simeq 56 \times 0.6 = 33.6$.

Solution *(using Madhava's table). Denote by θ the angle formed between the ground and the ramp. Then*

$$\sin\theta = \frac{45}{56} \simeq 0.8036.$$

From Madhava's table, you find that the closest value is $\sin 52.50 \simeq 0.7934$ *and so we set* $\theta \simeq 52.5$. *The complimentary angle is* $90 - \theta = 90 - 52.5 = 37.5$. *Again from Madhava's table* $\sin 37.50 \simeq 0.6087 \simeq \cos 52.50$ *so that* $x \simeq 56 \times 0.6087 = 34.0872$.

Using a calculator we may determine that $x \simeq \sqrt{56^2 - 45^2} \simeq 33.3317$. So P322 produces a more accurate solution.

Problem 2. *The side of the Great Pyramid at Giza had an original height of* 280 *cubits and a width of base of* 440 *cubits. What was the length z of an edge of the pyramid (from a corner to the top)?*

Since half of the base would be 220 cubits, we can verify that the *seqed* or *ukullû* of the side of the pyramid would have been 220 : 280, which gives indeed the famous value of $5\frac{1}{2} : 7$, or 5 palms and 2 fingers per cubit. But to get at the edge of the pyramid, we must use a triangle of height 280 and approximate base $220\sqrt{2}$.

Solution *(using Madhava's table). Madhava's table allows you to approximate* $\sqrt{2}$ *as* $2\sin 45 \simeq 2 \times 0.70710681 = 1.41421362$. *So the base of the right triangle is* $220 \times 1.41421362 = 311.1269964$ *and its height is* 280. *The ratio of these is* $\tan\theta \simeq \frac{280}{311.1269964} \simeq 0.9000$ *to four decimal places, where θ is the angle of inclination of the edge of the pyramid. Obviously some values of* tan *are needed, but since they are not directly on the table they need to be computed:* $\tan 45 = 1$ *is easy, while*

$$\tan 41.25 \simeq \frac{\sin 41.25}{\cos 41.25} \simeq \frac{\sin 41.25}{\sin 48.75} \simeq \frac{0.6593}{0.7518} \simeq 0.8770.$$

So taking $\theta = 41.25$ *you get that the required length is*

$$z \simeq \frac{280}{\sin 41.25} \simeq \frac{280}{0.6593} \simeq 424.6929.$$

Solution *(using P322). From an OB perspective, the right triangle formed by the corner, the center of the base, and the top of the pyramid ought to be considered to have a short side of* $b = 280$ *and a long side of* l, *which by the Diagonal rule in the horizontal isosceles triangle of side length* 220 *satisfies* $l^2 = (220)^2 \times 2 = 96\,800$. *Putting these values into the Diagonal rule now in the vertical triangle, the square of the diagonal is then* $d^2 = 96\,800 + 78\,400 = 175\,200$ *and hence you get a square ratio of*

$$\delta^2 = \frac{d^2}{l^2} = \frac{175\,200}{96\,800} = \frac{219}{121} = 1.8099.$$

The relevant row of P322(CR-Decimal8) is row 5 which is

$$|0.90277777|1.34722222|1.81500771|65|97|5|$$

and from which we can then use the integral values of $b_5 = 65$ *and* $d_5 = 97$ *to compare ratios*

$$\frac{z}{280} \simeq \frac{97}{65}$$

and so $z \simeq 417.8461$.

A more accurate modern answer correct to 8 decimal places is 418.568 990 73, so we see that the OB table is again the clear winner as far as accuracy is concerned. Note that the OB solution has avoided mention of any irrationalities, and it shows also that the mysterious column I' allows access to the table in a variety of important situations coming from the Diagonal rule, as it is a squared quantity! This solution also notably exhibits the utility of the entries b and d from columns II' and III', as the integers 65 and 97 there are both more accurate and generally easier to work with than the decimal numbers 0.90277 and 1.34722 in columns I and II.

Use of trigonometric tables is much improved if interpolation is available. We will see shortly, following Knuth (1972) that interpolation played a role in OB mathematics, so if we add that to the story then P322(CR-Decimal8) becomes a very formidable table.

To summarize: without interpolation, accuracy on trigonometric exercises is highly dependent on the actual values, and consequently how close we are to given rows of the tables; nevertheless in the majority of trigonometric problems that are phrased just in terms of lengths, P322 will beat Madhava's table. The reasons are three-fold. The first and simplest reason is that P322(CR-Decimal8) contains 38 rows compared to the 24 rows in Madhava's table. Secondly Madhava's table concentrates on values of sin, which are useful but limited (more modern tables will also have values of cos, tan and cot and possibly the other ratios as well), while the holistic approach of P322(CR) gives you the entire triangle, and you can choose which particular ratios or quantities you are interested in. Thirdly, the completely correct aspect of P322(CR) means that numerical calculations are precise; in contrast errors in our modern approximations are amplified by numerical calculations.

In addition, it turns out that when we move to more sophisticated applications of trigonometry, the exact sexagesimal trigonometry is both more general and powerful, with the laws of rational trigonometry being polynomial typically of degree two or three, and not requiring transcendental functions, as demonstrated by Wildberger (2005). Hence we see that within P322 there is a powerful alternative view of trigonometry based not on angles but on ratios of sides and squared quantities going back to OB times. No subsequent table, from Hipparchus to Madhava to al-Kashi to Rheticus to the monumental 18th century French Cadastre, can compete with P322 with regards to precision – P322 is unique as it contains the world's only exact trigonometric table.

6.3. The role of interpolation and approximation

The basic strategy in applying P322 is to simply find the row which best fits the given data through looking up one of the data entry values of β, δ, β^2 or δ^2. But part of the potential power of a trigonometric table is that it allows an able practitioner to also compare a given value with adjacent rows in the table, and to *interpolate data accordingly*. So a very pertinent question is: did the OB scribes have a system of interpolation?

In fact they did, as was demonstrated by the OB tablet AO 6770, which is in the Louvre (Neugebauer, 1935, 37; Thureau-Dangin, 1938, 71). We give Neugebauer's transcription of the second of the five problems given in this tablet, along with our (literal) translation of his German version into English.

The problem concerns an interest calculation, and it is significant for a number of reasons. Its importance in establishing that OB mathematics was capable of linear interpolation was pointed out by Knuth (1972, 675) – but his publication on OB algorithms in a computing machinery journal has perhaps been missed by some.

9 **a-ga-na** 1 (gur) gur *a-na si-pá-at i-di-in-ma*
10 *i-na ki ma-si ša-na-tim li-im-ta-ha-ra*
11 *at-ta i-na e-pe-ši-i-ka a-na* mu 4-kam ru(?) *-pi-is*
12 *an-nu-ú-um a-na* 2 (gur) gur *mi-nam im-tar* (?)
13 *mi-nam a-na sa i-na sa* mu 3-kam *i-li-ú uš-ta-ka-an*
14 *ša i-na li-ib-bi* mu 4-kam *a-na* 2 (gur) gur ba-zi
15 2.33.20 *i-na-ad-di-nam-ma*
16 *i-na li-ib-bi* mu 4-kam 2.33.20 ba-zi
17 mu KAK *ú* ud(*ûmû*) *i-na-ad-di-nam*

Here is our translation of Neugebauer's translation.

9 Now: 1 gur was invested
10 After how many years will it be equal?
11 You with your doing: for the 4th year find (?)
12 This for 2 gur, what is the excess?
13 What is it that for the 3rd year extended is, is set down?
14 What from the 4th year has 2 gur removed?
15 2.33.20 you get and
16 from 4 years 2.33.20 is removed
17 The full years and the day you get.

According to Neugebauer, the interest rate referred to in this problem is 20% per annum, which figures in other problems of the time. (This may also explain why the fraction 5/6 has a special role in OB mathematics, as seen in multiplication tables for it.)

The problem in the tablet asks: how long will it take to double your money (or grain) if you invest 1 gur at an annual rate of 20% interest? The method is the following. Find out how much more than 2 gur you would have after 4 years. Find out how much more you would have after 4 years than 3 years. Assuming that interest is accumulated linearly between the years, how many months (we are assuming 12 months in a year!) are needed to get from 2 gur to the 4 year total? Subtract this number from 4 years to get the answer.

There is a rather subtle interpolation problem here which we are being asked to solve. We can calculate, using our modern understanding but also with sexagesimal notation as follows: In 3 years the total

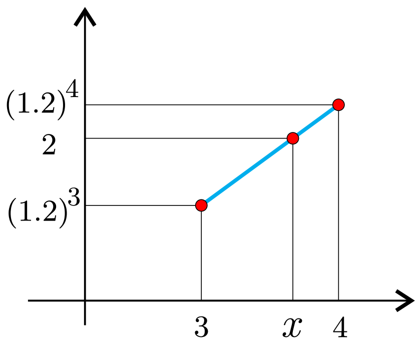

Figure 5.

is $(1;12)^3 = 1;43.40.48$, while in 4 years the total is $(1;12)^4 = 2;04.24.57.36$. So a total of 2 gur occurs somewhere in this interval. The ratio of $(1;12)^4 - 2 : (1;12)^4 - (1;12)^3$ is then $0;04.24.57.36 : 0;20.44.09.36$. Since all these are regular numbers, the OB scribe can make the calculation

$$\frac{(1;12)^4 - 2}{(1;12)^4 - (1;12)^3} \times 12 = \frac{0;04.24.57.36}{0;20.44.09.36} \times 12 = 2.33.20$$

where the multiplication by 12 converts from years to months. That is then how much less than 4 years we need to take to double our investment.

Implicit here are relations that math undergraduates would likely visualize geometrically as in Figure 5 (not to scale), and describe using slopes, through an equivalent decimal equation such as

$$\frac{(1.2)^4 - 2}{4 - x} = \frac{(1.2)^4 - (1.2)^3}{4 - 3}.$$

This is quite a sophisticated understanding. Knuth (1972, 675) concludes his discussion of this OB problem with the statement:

> This procedure suggests that the Babylonians were familiar with the idea of linear interpolation. Therefore the trigonometric tables in the famous “Plimpton tablet” ... were possibly used to obtain sines and cosines in a similar way.

If we interpret sines and cosines as transcendental functions of angles then his second suggestion is clearly anachronistic. But if we interpret them as simply ratios of sides in a right triangle, then Knuth’s remark is prescient.

We conclude our presentation of evidence of the power of P322(CD-Decimal8) by revisiting the calculation of the edge length z of the Great Pyramid.

Solution *(with interpolation). Recall that row 5 with* $\delta_5^2 = 1.81500$, $b_5 = 65$ *and* $d_5 = 97$ *gave us the estimate* $z \simeq 417.8461$. *Now this could be combined with row 6*

$$|0.88611|1.33611|1.78519|319|481|6|$$

to get also

$$\frac{z}{280} \simeq \frac{481}{319}$$

and so another estimate $z \simeq 422.1944$. *Perform a linear interpolation between the two estimates based on the original* δ^2 *values:*

$$\frac{417.8461 - 422.1944}{1.81500 - 1.78519} \simeq \frac{417.8461 - z}{1.81500 - 1.8099}$$

to get $z \simeq 418.5900$. *Comparing with the modern answer of* 418.56899073 *we see remarkable accuracy, in fact just using the original columns* I', II' *and* III' *of the extant tablet.*

7. Conclusion

Artifacts from ancient civilizations offer us a glimpse at their complex social and scientific achievements. Sophisticated artifacts offer us a larger window, and P322 is undeniably complex. Through P322 we see a forgotten ancient approach to trigonometry that is based on exact sexagesimal ratios.

The approximate nature of modern trigonometry is so culturally enshrined that we give it no second thought. It comes as a complete surprise, then, to find that the OB culture developed a trigonometry that actually only contains exact information, and so any imprecision is a consequence of using the table rather than inherent in the table itself.

P322 should be seen as an exact sexagesimal trigonometric table text, and we now recall all the evidence which points towards this conclusion: the arrangement of the rows ordered according to the ratios β, δ, β^2 and δ^2, the completeness with respect to rows which describe the entire breadth of possible shapes, the completeness with respect to columns which can be indexed by any ratio of sides, the ingenious way in which the squared quantities β^2 and δ^2 in column I' can be used to access the table, the simplified values b and d which have been optimized so the user can construct their own approximation to b/d or d/b, and the fact that P322 compares favorably with trigonometric tables from 3000 years later. No other explanation has achieved this level of cohesion with the evidence.

P322 is historically and mathematically significant because it is both the first trigonometric table and also the only trigonometric table that is precise. Irrational numbers and their approximations are seen as essential to classical metrical geometry, but here we have shown they are not actually necessary for trigonometry. If the dice of history had fallen a different way, and the deep mathematical understanding of the scribe who created P322 not been lost, then very possibly ratio-based trigonometry would have developed alongside our angle-based approach.

This new interpretation of P322 significantly elevates the status of Babylonian mathematics, and the vast number of untranslated tablets are likely to contain many more surprises waiting to be found. The discovery of trigonometry is attributed to the ancient Greeks, but this needs to be reconsidered in light of the much earlier, computationally simpler and more precise Babylonian style of exact sexagesimal trigonometry. In addition to being historically significant, P322 also brings the founding assumptions of our own mathematical culture into perspective. Perhaps this different and simpler way of thinking has the potential to unlock improvements in science, engineering, and mathematics education today.

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