KELLIE O. GUTMAN

Quando Che'l Cubo

In the history of mathematics, the story of the solution to the cubic equation is as convoluted as it is significant. When I first read an account of it in William Dunham's Journey Through Genius[1] *in 2000, I was captivated by the personalities, the intrigues, and the controversies that were part of mathematics in sixteenth-century Italy. For those unfamiliar with it, the story runs as follows:*

In the early 1500s, the mathematician Scipione del Ferro of the University of Bologna discovered how to solve a *depressed* cubic—one without its second-degree term—but in the style of the day he kept his discovery to himself. On his deathbed in 1526 he divulged the solution to his student Antonio Fior.[2]

Eight years later Niccolò Fontana, known as "Tartaglia" ("Stutterer"), hinted that he knew how to solve cubics that were missing their *linear* term. Fior publicly challenged Tartaglia to a contest in February of 1535, sending him a set of thirty depressed cubics to solve. At first Tartaglia was stumped, but with the deadline approaching, he figured out how to solve depressed cubics, thus winning the challenge.

In Milan, the mathematician/physician Gerolamo Cardano heard about Tartaglia's grand accomplishment. For several years, he pleaded with Tartaglia to tell him his secret. Finally in 1539, Tartaglia traveled to Milan from Venice and told Cardano the solution, but made him swear never to publish it.

With continued research, Cardano figured out how to reduce a general cubic to a depressed one, thus completely solving the classical problem of the cubic. Then his assistant Lodovico Ferrari extended this string of discoveries by solving fourth-degree problems, but both men refrained from publishing their results because they were based on Tartaglia's solution.

On a hunch, Cardano and Ferrari traveled to Bologna in 1543 to look at the papers of Fior's master, Scipione del Ferro, who they must have reasoned also knew the solution to depressed cubics. They found Scipione's original algorithm and it was identical to Tartaglia's.

Finally, Cardano felt released from his oath to Tartaglia. Giving full credit to both Scipione and Tartaglia, he published the solution to the depressed cubic, his own solution to the general cubic, and Ferrari's solution to the quartic, in 1545, in a huge tome, *Ars Magna*. This widely dispersed work is considered by many to be the first book ever written entirely about algebra. In it, Cardano devoted little space to the solution of the quartic, because a fourth power was considered a meaningless concept, not corresponding to any physical object.

Tartaglia was enraged. The following year, in his own book *Quesiti et inventioni diverse*, Tartaglia presented his version of a long conversation between himself and Cardano from their encounters six years earlier, in which he made it clear that his "invention" was not to be disclosed. He then presented his solution in a poem, saying this was the easiest way for him to remember it.

* * *

[1]Dunham, William. *Journey Through Genius*. New York: John Wiley & Sons, Inc., 1990.

[2]Many of the historical facts came from the MacTutor History of Mathematics archive of the School of Mathematics and Statistics, University of St Andrews, Scotland Created by John J. O'Connor and Edmund F. Robertson
http://www-history.mcs.st-andrews.ac.uk/history/index.html

Quando che'l cubo[3]

Quando che'l cubo con le cose appresso
Se agguaglia à qualche numero discreto
Trovar dui altri differenti in esso.

Dapoi terrai questo per consueto
Che'l lor produtto sempre sia eguale
Al terzo cubo delle cose neto,

El residuo poi suo generale
Delli lor lati cubi ben sottrati
Varra la tua cosa principale.

In el secondo de cotesti atti
Quando che'l cubo restasse lui solo
Tu osservarai quest'altri contratti,

Del numer farai due tal part'à volo
Che l'una in l'altra si produca schietto
El terzo cubo delle cose in siolo

Delle qual poi, per commun precetto
Torrai li lati cubi insieme gionti
Et cotal somma sara il tuo concetto.

El terzo poi de questi nostri conti
Se solve col secondo se ben guardi
Che per natura son quasi congionti.

Questi trovai, e non con passi tardi
Nel mille cinquecentè, quatro e trenta
Con fondamenti ben sald'è gagliardi

Nella citta dal mar' intorno centa.

Any Italian who encountered this poem would have immediately recognized it as being written in the celebrated form known as *terza rima*, invented by Dante Alighieri and used in his masterwork, *La Divina Commedia.* Like Dante, Tartaglia wrote in Italian, which was the language of literature, not Latin, which was the main language of science: this was because Tartaglia did not know Latin. *Terza rima* is made up of eleven-syllable, or hendecasyllabic, lines. Each line is iambic with five stressed and six unstressed syllables. It is an especially fitting form for a poem about cubic equations because there are two sets of threes contained in it: the poem is written in tercets, or three-line stanzas, and all the rhymes, except at the start and finish of the poem, come in triplicate, with the center line of each tercet rhyming with the outer lines of the tercet following it, thus propelling the poem forward. This form is extraordinarily well-known by Italians.

* * *

In the early sixteenth century, algebra was rhetorical—that is, variables, the equal sign, negative numbers, and the concept of setting something equal to zero did not exist. Everything was described solely through words. Instead of writing "$x^3 + mx = n$" one would write *cubo con cosa agguaglia ad un numero* or "cube and thing are equal to a number." It was a cumbersome system, and calculations and proofs were difficult to follow.

When I saw Tartaglia's poem for the first time in early 2004, I was so taken with it that I had to translate it, but I soon found myself faced with a dilemma. Either I could translate it literally as he wrote it, and have it be as obscure as his was (and it *is* obscure), or I could do a modern translation and essentially say, "This is what he meant, though it is not what he said." The second way would make it very clear for today's reader. Neither of these felt quite right to me. Instead, I decided to bridge the two worlds of Renaissance mathematics and modern mathematics, attempting to retain the poem's ancient flavor along with its *terza rima*, but using variables where Tartaglia used only words.

Because the vast majority of Italian words end in an unstressed syllable, it is natural to have iambic lines of poetry with eleven syllables. It is slightly more difficult in English. In my translation I have used an alternating pattern of masculine rhymes, with the stress and rhyme on the final syllable, and feminine rhymes, which rhyme on the stressed penultimate syllable.

* * *

When X Cubed

When x cubed's summed with m times x and then
Set equal to some number, a relation
Is found where r less s will equal n.

Now multiply these terms. This combination
rs will equal m thirds to the third;
This gives us a quadratic situation,

Where r and s involve the same square surd.
Their cube roots must be taken; then subtracting
Them gives you x; your answer's been inferred.

The second case we'll set about enacting
Has x cubed on the left side all alone.
The same relationships, the same extracting:

[3]Tartaglia, Niccolò, Quesiti et inventioni diverse de Niccolò Tartalea Brisciano.
[Stampata in Venetia per Venturo Roffinelli, 1546.]
Quesito XXXIIII. Fatto personalmente dalla eccellentia del medesimo messer Hieronimo Cardano in Millano in casa sua adi. 25. Marzo.1539
"Quando chel cubo con le cose apresso . . . "—begins leaf 123 recto
" . . . Nella citta dal mar'intorno centa."—ends leaf 123 verso
(Also reproduced on the following Web site:
http://digilander.libero.it/basecinque/tartaglia/equacubica.htm)

Seek numbers r and s, where the unknown
  rs will equal m-on-3 cubed nicely,
  And summing r and s gives n, as shown.

Once more the cube roots must be found concisely
  Of our two newfound terms, both r and s,
  And when we add these roots, there's x precisely.

The final case is easy to assess:
  Look closely at the second case I mention—
  It's so alike that I shall not digress.

These things I've quickly found, they're my invention,
  In this year fifteen hundred thirty-four,
  While working hard and paying close attention,

Surrounded by canals that lap the shore.

So what exactly is Tartaglia saying? He's saying that when $x^3 + mx = n$, two other numbers, r and s, can be found such that $r - s = n$ and $rs = (m/3)^3$. Mathematicians of his day knew that when they were told the values of a product and a difference (or sum) of two unknown numbers, they had what I have called a "quadratic situation" (there was no such thing as a quadratic equation). They had an algorithm, which was tricky but manageable, to find the solutions to such situations. In fact, because they didn't recognize negative numbers, they had a set of variants of what we would think of as one single thing, namely the quadratic formula. Using the applicable variant, one could solve for r and s. Next, Tartaglia is telling his readers to take the cube roots of the numbers r and s, and to subtract the cube root of s from that of r. This will be x, the solution to the given cubic.

He then moves on, in the fourth stanza, to what was considered a different situation, when $x^3 = mx + n$, and he gives the solution again. The third case, when $x^3 + n = mx$, he says, in the seventh stanza, is almost exactly like the second, and so he leaves that for the reader to figure out. He concludes with a flourish by claiming credit for the discovery, and telling his readers he found the solution in Venice.

* * *

Tartaglia discovered his solution by thinking about an actual physical cube. To him, and most likely to Scipione as well, the solution to a problem involving a cubic was embodied in a real cube. Seven hundred years earlier, in Baghdad, Al-Khwarizmi (from whose name comes the word "algorithm") thought about a square when working on problems involving quadratics. He came up with a formula for "completing the square" to solve such problems.

An equation of the type $x^2 + mx = n$ can be pictured by first drawing a square of side x (see Figure 1). Next make two congruent rectangles of length x and width $m/2$, and attach them to two adjacent sides of the square. The dimensions $m/2$ and x are picked for very good reasons—two rectangles of this size together make up an area of mx, to add to the original square of the area x^2, and these three together have a joint area of n, giving $x^2 + mx = n$.

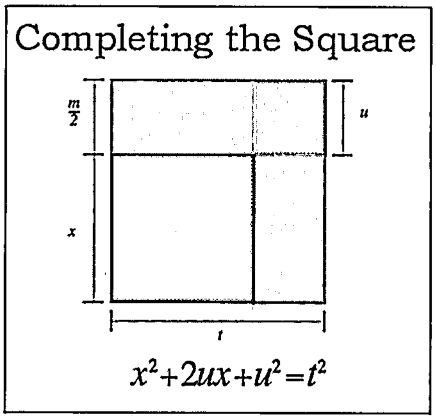

Figure 1. A version of Al-Khwarizmi's completion of the square. Moving left to right, the equation can be read directly off the diagram.

The picture looks like a square cardboard box from above, with two adjacent flaps open. It calls out for one other square, of side length $m/2$, to be drawn in, in order to *complete* the larger square. Let's call the side of this new big square t, and the side of the new little square u. When we combine the area n with the area u^2, which is $(m/2)^2$, we get the area of the larger square, t^2. The square root of this square area—that is, the square root of $n + (m/2)^2$—gives us the side length t. But t is equal to $x + m/2$, so x equals $\sqrt{n + (m/2)^2} - m/2$. Thus by completing the square, Al-Khwarizmi solved the quadratic.

In a similar fashion to Al-Khwarizmi, Tartaglia envisioned "completing the cube" to solve the depressed cubic. He took Al-Khwarizmi's drawing into a third dimension (Fig. 2).

With an equation of the form $x^3 + mx = n$, he started by imagining a cube of side x (this corresponded to the square of side x in two dimensions). He then looked for analogous volumes to play the role of the two rectangles flanking the square of side x, but since he was in three dimensions he instead imagined three slabs. Each had one side of length x, and two other sides of unknown lengths, which we will call t and u. These three slabs fit neatly

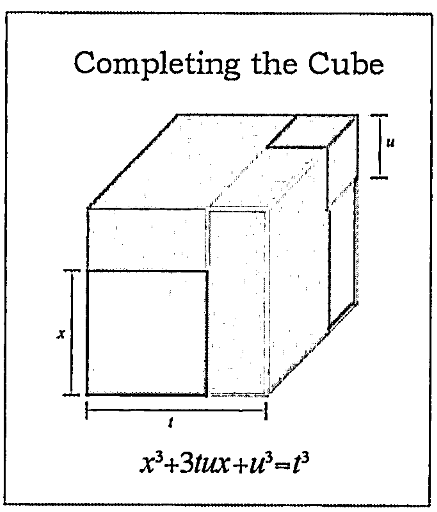

Figure 2. Tartaglia's completion of the cube. Once again the equation can be read directly off the diagram.

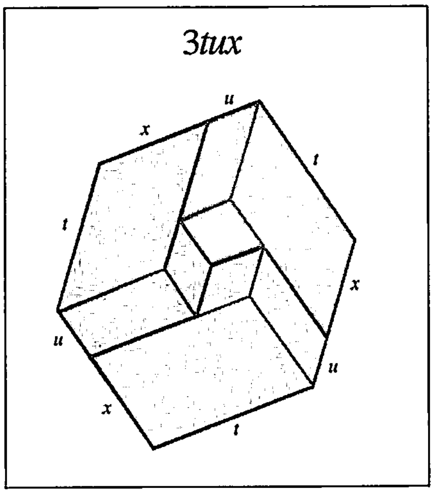

Figure 3. Like a Necker cube, this picture flips between two interpretations. In the intended interpretation, one sees three slabs, each of volume *tux*, swirling counter-clockwise around a (missing) cube of side *u*. In the other interpretation (and this came as a complete and lovely surprise to me) one sees a cube of side *u* sitting nestled in one corner of a cutaway cube of side *t*, and thanks to the colors painted on the large cube's walls, one cannot help "seeing" (though they are missing) the three slabs of volume *tux*, once again swirling counter-clockwise about the little cube of side *u*.

around the cube of side x, thus giving him a larger cube of side t, but (as before) with one crucial piece missing. In order to *complete* the larger cube, Tartaglia added one last cube of side u (corresponding to the little square of side u that completed Al-Khwarizmi's square; Fig. 3).

Each of the three slabs has sides of length t, u, and x, and so the total volume of the slabs is $3tux$. Now the volumes of the two interior cubes are x^3 and u^3, so the total volume of the big cube is $x^3 + 3tux + u^3$, but of course it is also t^3. In symbols,

$$x^3 + 3tux + u^3 = t^3.$$

We can imagine Tartaglia striving to imagine the dimensions of a physical cube that would represent the solution to an actual depressed-cubic problem posed by his challenger Fior. In Al-Khwarizmi's quadratic, the value of u is known instantly without calculation. But in the case of the cubic, things are not so simple, because one doesn't know the value of either t or u. In the realm of all possible cubes, Tartaglia needed to find the one cube with the exact dimensions that satisfy his problem. He had to imagine the lengths u and t both changing (the overall cube growing and shrinking, and also the cube of side x changing size because it is determined by t and u, its side being $t - u$). It seemed as if the search for the proper cube could only be carried out by trial and error, without any formula, and thus it was not really a mathematical solution.

At this point, though, rather than giving up, Tartaglia has a brilliant insight. Looking at his equation (above), he realizes that if he merely moves u^3 to the right side, it will give him a new equation that precisely embodies Fior's depressed cubic $x^3 + mx = n$, with $3tu$ playing the role of m and $t^3 - u^3$ playing the role of n.

$$\begin{array}{ccccc} x^3 + & 3tux & = & t^3 - u^3 \\ & | & & | \\ & | & & | \\ x^3 + & mx & = & n \end{array}$$

This is a breakthrough moment for Tartaglia, because it tightly connects the unknowns, t and u, with the knowns, m and n:

$$3tu = m, \qquad t^3 - u^3 = n.$$

This is very promising, but he is not there yet, because he doesn't know how to solve these equations for t and u in terms of m and n. As he considers these equations, however, Tartaglia sees that he has a situation that comes very close to being a quadratic in t and u, but just misses—namely, he has a product and a difference involving t and u, but one of them involves their cubes. Thus provoked, Tartaglia has another insight. He gives names to the two cubic volumes, calling t^3 "r" and u^3 "s," knowing that in this way he will obtain a *genuine* quadratic situation (involving a difference and a product) with his new variables r and s. Now his equations are

$$r - s = n$$
$$rs = (m/3)^3.$$

The last equation is an immediate consequence of the definition of r and s. From $3tu = m$ it follows that $tu = m/3$, and thus, cubing both sides, $t^3u^3 = (m/3)^3$.

Now he is operating in familiar territory. He can easily find his quadratic by eliminating r as follows: $r = n + s$ and therefore $rs = s(n + s)$, giving

$$s^2 + ns = (m/3)^3.$$

Tartaglia has at last come full circle. After starting out with Al-Khwarizmi's model of completing the square in order to come up with his own model of the cubic, he now applies Al-Khwarizmi's square-completing method to solve this quadratic for r and s; having gotten those, he can then take their cube roots to obtain the values of t and u. Then he merely subtracts u from t, and x has been found.

* * *

When Cardano published *Ars Magna*, rather than giving a general proof, he illustrated the solution to this particular cubic: $x^3 + 6x = 20$. Following the poem's directions, here is how it is solved.

$$x^3 + 6x = 20$$
$$r - s = 20$$
$$rs = (6/3)^3 = 2^3 = 8$$
$$r = 20 + s \text{ and therefore } s(20 + s) = 8$$
$$s^2 + 20s = 8$$
$$s^2 + 20s - 8 = 0.$$

Using the quadratic formula to solve for s, we get

$$\begin{aligned} s &= (-20 \pm \sqrt{400 + 32})/2 \\ &= -10 \pm \sqrt{108} \\ &= \sqrt{108} - 10 \\ r &= s + 20 = \sqrt{108} + 10. \end{aligned}$$

Numerically,

$$r = 20.3923 \text{ and } s = .3923.$$

Then, taking these numbers' cube roots,

$$x = \sqrt[3]{r} - \sqrt[3]{s}$$
$$x = 2.73205 - .73205$$
$$x = 2.$$

If we plug this back into the original equation $x^3 + 6x = 20$, we find that it is correct: $8 + 12 = 20$. The method works, although it must be admitted that it makes it look fortuitous that the answer is a simple integer.

* * *

Finding a solution by radicals to the cubic was a monumental accomplishment. However, it led to a thorny obstacle: in the case of a cubic equation that had only *one* real root (back then, mathematicians would have said the equation had only one root *at all*, for no one suspected that all cubics have three roots), the algorithm always yielded that root. By contrast, in the case of a cubic that had *three* real roots, the algorithm seemed to yield nonsense. Even if the three real roots were already known, it led to expressions featuring negative numbers under the square-root sign, a situation that Cardano dubbed the *casus irreducibilis*, reflecting the fact that Renaissance mathematicians were not comfortable with negative numbers, let alone their square roots.

The Bologna mathematician Rafael Bombelli took Cardano's *casus irreducibilis* very seriously and tried to make sense of the square roots of negative numbers. He figured out how to do the four standard arithmetical operations not only with negative numbers but also with their "imaginary" square roots, and shortly before his death in 1572, he published a book on this topic titled *Algebra*, in which he presented an early symbolic notation system. Although he never found out how to take cube roots of complex numbers in general, he was able to determine the complex cube root called for by Cardano's algorithm in one specific case, and he showed that the two imaginary contributions to the final answer canceled each other out, leading to a purely real root. More details of Bombelli's work will be found in a recent scholarly article in this journal by Federica LaNave and Barry Mazur; see vol. 24, no. 1 (2002), 12–21.

Despite this accomplishment, Cardano's formula provided Bombelli with only one of the equation's three roots, and it took another 40 years until François Viète figured out how to find the other two real roots, and then a further 300 years until mathematicians penetrated the mystery of the *casus irreducibilis* and finally understood why complex numbers were needed to express the real roots to cubic equations through radicals.

When Ferrari based his solution of the quartic equation on that of the cubic, just as Tartaglia had based his solution of the cubic on that of the quadratic, it seemed as if this clever method could go on indefinitely: lower the degree of an equation by one, and use this new equation's formula to help solve the original. But when mathematicians tried to solve the quintic equation in this way, they hit a brick wall. It wouldn't yield.

For the next 250 years, mathematicians struggled to solve quintics by radicals. Finally in 1799, Paolo Ruffini, another mathematician/physician, wrote a book *Teoria Generale delle Equazioni*, offering a proof that fifth-degree equations—indeed, all equations of degree greater than four—were in general unsolvable by radicals; but almost no one accepted his claims. Twenty-two years later the distinguished French mathematician Cauchy wrote to Ruffini, praising his proof, but few people agreed with Cauchy. In a few years, however, Niels Henrik Abel in 1825 and Evariste Galois in 1830 published works on the unsolvability of the quintic equation and equations of higher order, and their discoveries, which were centered on the symmetry groups of the roots, were widely accepted.

For the thousand or so years between the destruction of the Library of Alexandria and the Renaissance, European mathematics, with a few notable exceptions, had made slow progress. But the Italian mathematicians who worked on solving the cubic initiated a series of events that led to the use of negative numbers, complex numbers, powers and dimensions higher than the third, and symbolic algebra, with its highly efficient system of symbol manipulation. This work, spanning roughly one hundred years, reinvigorated mathematics and led directly to many of the discoveries of the modern era.

AUTHOR

KELLIE O. GUTMAN
75 Gardner Street
West Roxbury, MA 02132-4925
USA
e-mail: RKLGutman@comcast.net

Kellie Gutman has studied mathematics and audiology—and, since 1999, poetry. One piece of mathematical research she wrenched into poetic form, quite impressively; see *The Mathematical Intelligencer* 23 (2001), no. 3, 50. With her husband, Richard Gutman, she is co-owner for 25 years of a company specializing in audio-visual presentations for museum installations; co-author of two books; and parent of Lucy.