Reading Bombelli's x-Purgated Algebra

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Maxim Bruckheimer received his B.A. and Ph.D. (in global differential geometry) from Southampton University in the U.K. His interest in mathematics education started at that time and he coauthored many books for students and teachers. After teaching mathematics at the City University in London, he helped found the Open University in 1969. As Dean of the Mathematics Faculty, he was responsible for the first multimedia courses which established the University's reputation. In 1974 he moved to the Weizmann Institute of Science in Israel, where he heads the mathematics group in the Department of Science Teaching. A major interest is the history of mathematics in mathematics education. He and Abraham Arcavi have produced historical worksheet materials for teachers and are now working on a set of illustrated historical activities for use in the junior high school classroom.

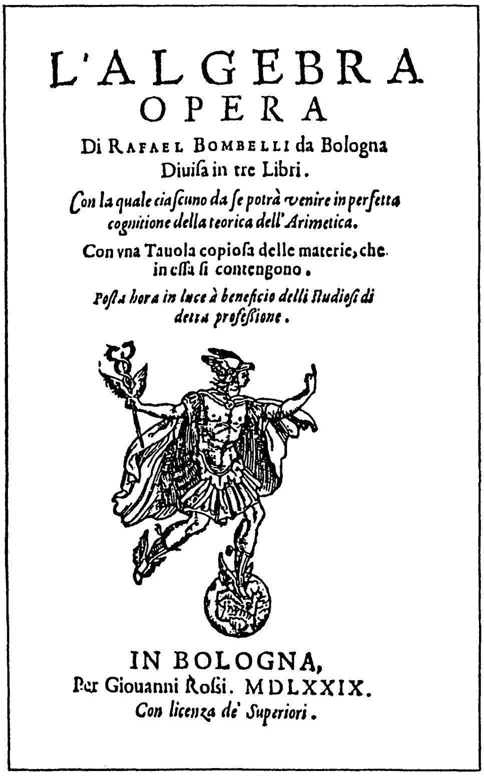
Reading mathematics is hard work and reading a four hundred year old mathematics text is four hundred times harder. The language, notation and also the spirit are different from ours. If the reader is not already convinced from past experience, the following extract should prove the point.

> *Let us first assume that if we wish to find the approximate root of 13 that this will be 3 with 4 left over. This remainder should be divided by 6 (double the 3 given above) which gives $\frac{2}{3}$. This is the first fraction which is to be added to the 3, making $3\frac{2}{3}$ which is the approximate root of 13. Since the square of this number is $13\frac{4}{9}$, it is $\frac{4}{9}$ too large, and if one wishes a closer approximation, the 6 which is the double of the 3 should be added to the fraction $\frac{2}{3}$, giving $6\frac{2}{3}$, and this number should be divided into the 4 which is the difference between 13 and 9,...*

If you understood—don't read on. If you didn't, then in this article we illustrate in some detail how a little perseverance can turn "obscurity" into a rewarding experience for students.

Bombelli's Method

The text quoted above was written by Rafael Bombelli, a 16th century Italian mathematician. He wrote an important textbook which appeared in two editions, *L'algebra parte maggiore dell'arithmetica* (1572) and *L'algebra* (1579). It includes Bombelli's contributions to the solution of cubic and quartic equations and many geometrical problems solved algebraically (see, for example, [**2**] and [**4**]).



L'ALGEBRA
OPERA
Di Rafael Bombelli da Bologna
Diuisa in tre Libri.
Con la quale ciascuno da se potrà venire in perfetta cognitione della teorica dell'Arimetica.
Con vna Tauola copiosa delle materie, che in essa si contengono.
Posta hora in luce à beneficio delli studiosi di detta professione.
IN BOLOGNA,
Per Giouanni Rossi. MDLXXIX.
Con licenza de' Superiori.

Figure 1

In this paper we reproduce from the 1579 edition, the algorithm he introduced to approximate square roots, which is the subject of the obscure paragraph quoted in our introduction. At the time Bombelli wrote his *Algebra*, decimal fractions were not yet in use; they were introduced by Simon Stevin in his book *La disme* ("The tenth") in 1585. Bombelli developed his method using common fractions. We will follow in his footsteps. The chapter starts (the English version from which we quote is [**5**, p. 81]) with a very human touch—not exactly in the style of a modern textbook—serving precisely the purpose of engaging and motivating readers.

Method of Forming Fractions in the Extraction of Roots
(*Modo di formare il rotto nella estratione delle Radici quadrate*)

Many methods of forming fractions have been given in the works of other authors; the one attacking and accusing another without due cause (in my opinion) for they are all looking to the same end. It is indeed true that one method may be briefer than *another, but it is enough that all are at hand and the one that is the most easy will without doubt be accepted by men and be put in use without casting aspersions on another method. ... In short, I shall set forth the method which is the most pleasing to me today and it will rest in men's judgement to appraise what they see: meanwhile I shall continue my discourse going now to the discussion itself.*

Modo di formare il rotto nella eſtrattione delle Radici quadrate.

Molti modi ſono ſtati ſcritti da gli altri autori de l'vſo di formare il rotto;l'uno taſſando,e accuſando l'altro (al mio giudicio) ſenza alcun propoſito, perche tutti mirano ad un fine; E ben vero che l'una è più breue dell'altra, ma baſta che tutte ſuppliſcono, e quella ch è più facile, non è dubbio ch'eſſa ſarà accettata da gli huomini, e ſarà poſta in uſo ſenza taſſare alcuno; perche potria eſſere, che hoggi io inſegnaſſi una regola,laquale piacerebbe più dell'altre date per il paſſato, e poi veniſſe un'altro, e ne trouaſſe una più vaga, e facile, e coſi ſarebbe all'hora quella accetata, e la mia confutata,perche(come ſi dice)la eſperienza ci è maeſtra, e l'opra loda l'artefice. Però metterò quella che più à me piace per hora, e ſarà in arbitrio de gli huomini pigliare qual vorranno: dunque venendo al fatto dico. Che preſupoſto, che ſi voglia il proſſimo lato di 13, che ſarà 3, e auanzerà 4, il quale ſi partirà per 6 (doppio del 3 ſudetto) ne uiene $\frac{2}{3}$, e queſto è il primo rotto, che ſi hà da giongere al 3, che fà 3 $\frac{2}{3}$, ch è il proſſimo lato di 13, perche il ſuo quadrato è 13 $\frac{4}{9}$, ch'è ſuperfluo $\frac{4}{9}$, ma uolendoſi più approſſimare, al 6.doppio del 3 ſe gli aggionga il rotto,cioè li $\frac{2}{3}$, e farà 6 $\frac{2}{3}$, e per eſſo partendoſi il 4, che auanza dal 9 ſino al 13,

Figure 2

The above extract is followed by the text we quoted in our introduction. Even if we translate the "recipe" for extracting square roots described there from its rhetorical form (see, for example, [**6**, pp. 378–379]) into modern symbols, and follow the calculations, the procedure is still unmotivated. Why should one divide the remainder by 6? Why should one add the 6 to the $\frac{2}{3}$? Why should one divide it into 4? And so on.

Why did Bombelli find this method *the most pleasing*? It would seem to be hard to agree with him, at least when one reads it for the first time. However, in a subsequent paragraph, Bombelli himself provides a fuller explanation of his method, which we reproduce (the English version we quote here is taken from [**3**, pp. 69–70].), side by side with the modern notation.

1. *Let us suppose we are required to find the root of 13. The nearest square is 9, which has root 3. I let the approximate root of 13 be 3 plus 1 tanto* [unknown].	$3+x=\sqrt{13}$
2. *Its square is 9 plus 6 tanti p. 1 power. We set this equal to 13.*	$(3+x)^2$ $=9+6x+x^2=13$
3. *Subtracting 9 from either side of the equation we are left with 4 equal to 6 tanti plus one power.*	$6x+x^2=4$
4. *Many people have neglected the power and merely set 6 tanti, equal to 4. The tanto then comes to* $\frac{2}{3}\ldots$	$6x=4$ $\Rightarrow x=\frac{2}{3}$
5. *. . . and the approximate value of the root is* $3\frac{2}{3}$ *since it has been set equal to 3 p. 1 tanto.*	$\sqrt{13}\approx 3+x\approx 3\frac{2}{3}$.

So far, Bombelli has found a first approximation by "neglecting" (*lasciato andare*) the value of x^2 and thus obtaining $x=\frac{2}{3}$, the fraction that is to be added to 3 to obtain an approximation to $\sqrt{13}$. The second approximation is now found by taking into account what was neglected in the first approximation.

6. *However, taking the power into account, if the tanto is equal to* $\frac{2}{3}$, *the power will be* $\frac{2}{3}$ *of a tanto, which, added to the 6 tanti, will give us 6 and* $\frac{2}{3}$ *tanti, which are equal to 4.*	$6x+x^2=4$ $6x+xx=4$ $\downarrow$ $6x+\frac{2}{3}x=4$
7. *So the tanto will be equal to* $\frac{3}{5}$, *and since the approximate is 3 p. 1 tanto it comes to* $3\frac{3}{5}$.	$\Rightarrow x=\dfrac{4}{6+\frac{2}{3}}=\frac{3}{5}$ $\Rightarrow 3+x=3\frac{3}{5}$.

Note that Bombelli has taken the product $x^2=xx$, and given one x the value previously obtained, and the other becomes the new fraction that is to be added to 3 to obtain the next approximation to $\sqrt{13}$. He now uses this *double-entendre* for x recursively.

8. *But if the tanto is equal to* $\frac{3}{5}$, *the power will be* $\frac{3}{5}$ *of a tanto and we obtain* $6\frac{3}{5}$ *tanti equal to 4 . . .*	$6x+x^2=4$ $6x+xx=4$ $\downarrow$ $6x+\frac{3}{5}x=4\ldots$

Solving for x, Bombelli obtains a third approximation, $3\frac{20}{33}$.

The process can be continued for as long as one has patience. Or to put it in Bombelli's words: *e cosi procedendo si puo approssimare a una cosa insensibile* (and this process may be carried to within an imperceptible difference [**5**]).

Bombelli's Method and Continued Fractions

Bombelli's algorithm can also be described in the following way. We wish to find an x so that $\sqrt{13} = 3 + x$. Squaring both sides, $13 = (3+x)^2$. Whence $6x + x^2 = 4$ or $x(6+x) = 4$, and finally $x = 4/(6+x)$.

Note that in the expression $4/(6+x)$, the x takes the value obtained in the previous stage. However, the x on the left takes a new value obtained in the present stage. This corresponds to the *double-entendre* noted above. In more precise modern notation we would write

$$x_{n+1} = \frac{4}{6 + x_n}.$$

Using our notation we obtain the continued fractions

$$x_2 = \frac{4}{6 + x_1}, \qquad x_3 = \frac{4}{6 + x_2} = \cfrac{4}{6 + \cfrac{4}{6 + x_1}}, \qquad x_4 = \frac{4}{6 + x_3} = \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6 + x_1}}}.$$

One cannot assert that Bombelli invented continued fractions, since this form—or anything that could suggest it—does not appear in his text. However, we read (in [**6**, pp. 419–420]) that: "Although the Greek use of continued fractions in the case of greatest common measure was well known in the Middle Ages, the modern theory of the subject may be said to have begun with Bombelli (1572). ... The next writer to consider these fractions, and the first to write them in substantially the modern form was Cataldi (1613), and to him is commonly assigned the invention of the theory. His method was substantially the same as Bombelli's, but he wrote the result of the square root of 18 in the following form:

$$4.\&\,\frac{2}{8.\;\&\,\frac{2}{8.\;\&\,\frac{2}{8}}}\quad ."$$

Bombelli's Method in the Classroom

In this section we would like to suggest how Bombelli's method can be used in the classroom.

"Dictionary" questions. This activity helps the students to become acquainted with unknown notation, symbols, names of concepts, or formulations in the source.

For example, the following dictionary is to be completed by the student, as a first step in deciphering the text.

Rhetorical language of Bombelli	*Modern notation*
p. or plus	—
equal to	—
one quantity (unknown)	—
power (second, of unknown)	—
—	$3+x$
9 plus 6 *tanti* p. 1 power	—

The translation process illustrates the immense power which that little x, which we take so for granted, gives us. This is further illustrated in the next section.

"Redoing" and applying the mathematics. Once the terminology is understood, we give students the left side of the eight steps from the section "Bombelli's Method" and ask them to translate each step into modern notation. When the method is clear from the algorithmic point of view, we ask students to apply it to other numbers and thus give the experience some permanence. For example, the calculation for $\sqrt{2}$ gives the continued fraction

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}.$$

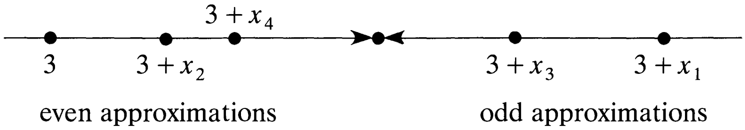
This activity can be integrated into one or more places in the curriculum. The main subject is obviously approximating square roots and Bombelli's method can be compared to others. It can also be discussed when dealing with fractions and continued fractions, as part of a unit on irrational numbers, or as a nice illustration of iteration.

Issues for discussion. We have overcome our perplexity when we first read Bombelli's "recipe." We take the opportunity to point out to students the moral learned from the exercise. Unlike reading in many other areas, reading mathematics, even when the notation is modern, also involves writing, redoing in other ways, drawing diagrams, and definitely rereading. But, the task is still incomplete. Bombelli's text raises many mathematical issues for the critical reader. We deal with some of them in the classroom in the following way.

Compare the successive approximations of $\sqrt{13}$. Resorting to calculators in order not to place the burden of the activity on comparing common fractions, we find that the first approximations in decimal form are:

$$3+x_1=3.66666\ldots,\quad 3+x_2=3.60,\quad 3+x_3=3.606060\ldots,$$
$$3+x_4=3.6055045\ldots.$$

Looking at the first four decimal digits of $\sqrt{13}=3.6055513\ldots$, one notices that the even approximations are less than $\sqrt{13}$ and increasing, and the odd approximations are greater than $\sqrt{13}$ and decreasing, as illustrated in the following diagram.



Had Bombelli presented his method today, he would immediately be required to *prove* that, if one continues the process indefinitely, the sequence will indeed converge to the desired root. A sketch of the proof follows.

The first approximation to $\sqrt{13}$ is greater than $\sqrt{13}$.

$$\left(3+\frac{2}{3}\right)^2 = 9+4+\frac{4}{9} > 13.$$

The second approximation to $\sqrt{13}$ is less than $\sqrt{13}$.

$$\left(3+\frac{3}{5}\right)^2 = 9+\frac{18}{5}+\frac{9}{25} < 13.$$

With mathematical induction, one can prove that every odd approximation is greater than $\sqrt{13}$ and that every even approximation is less than $\sqrt{13}$. The two proofs are similar. We sketch the case for the odd approximations. The induction starts with the step 1 above. Then we assume that $x_{2n-1} > \sqrt{13}-3$ and prove that x_{2n+1}, the subsequent odd approximation, is also greater than $\sqrt{13}-3$. In order to do that, we express x_{2n+1} in terms of x_{2n-1},

$$x_{2n+1} = \cfrac{4}{6+\cfrac{4}{6+x_{2n-1}}} > \cfrac{4}{6+\cfrac{4}{6+\sqrt{13}-3}} = -3+\sqrt{13}.$$

Next we show that every odd (even) approximation is less than (greater than) its predecessor. For example, in the case of the odd approximations, we have to show that

$$x_{2n-1} > x_{2n+1} \text{ or } x_{2n-1} - x_{2n+1} > 0.$$

We again express x_{2n+1} in terms of x_{2n-1} and, as before, the rest is algebra.

Finally, since the odd (even) approximations are decreasing (increasing) and always greater (less) than $\sqrt{13}$, as shown above, they must approach some limit. We still need to show that the limit l_o of the sequence of odd approximations is the same as the limit l_e of the sequence of even approximations.

One way to show that each limit is $\sqrt{13}$ is the following. Since

$$x_{2n+2} = \cfrac{4}{6+\cfrac{4}{6+x_{2n}}}$$

we can take the limits of both sides to obtain

$$l_e = \frac{4}{6 + \frac{4}{6 + l_e}}.$$

Solving the equation we obtain $l_e = \sqrt{13} - 3$. We obtain an identical result for odd approximations, and thus both limits coincide. Bombelli might have provided this argument, making use of his technique of *double-entendre*, had anyone in the 16th century thought that these things needed proving.

Final Comments

We used this activity with secondary mathematics teachers as a part of a sequence of activities on the historical development of irrational numbers [**1**]. We found that original sources are a very appropriate way to convey the feeling of mathematics as a living and developing human endeavor. The rhetorical or quasi-rhetorical expositions, in which the author shows personal preferences (*I shall set forth the method which is the most pleasing to me today*), and presents some arguments in a non-rigorous way (*Many people have neglected the power ...*) were very motivating. And the process of understanding the original source, first at the algorithmic level and then by discussing its mathematical validity as understood with modern eyes, was very enlightening. The teachers found that reading mathematics can be an engaging and enjoyable activity.

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The Best Laid Plans

"My idea then was to get through the course, secure a detail for a few years as assistant professor of mathematics at the Academy, and afterwards obtain a permanent position as professor in some respectable college; but circumstances always did shape my course different from my plans."

The Personal Memoirs of U.S. Grant.
Contributed by Mark Meyerson, Annapolis, MD.